Incorporating Tables into Proofs

by Dale Miller & Vivek Nigam

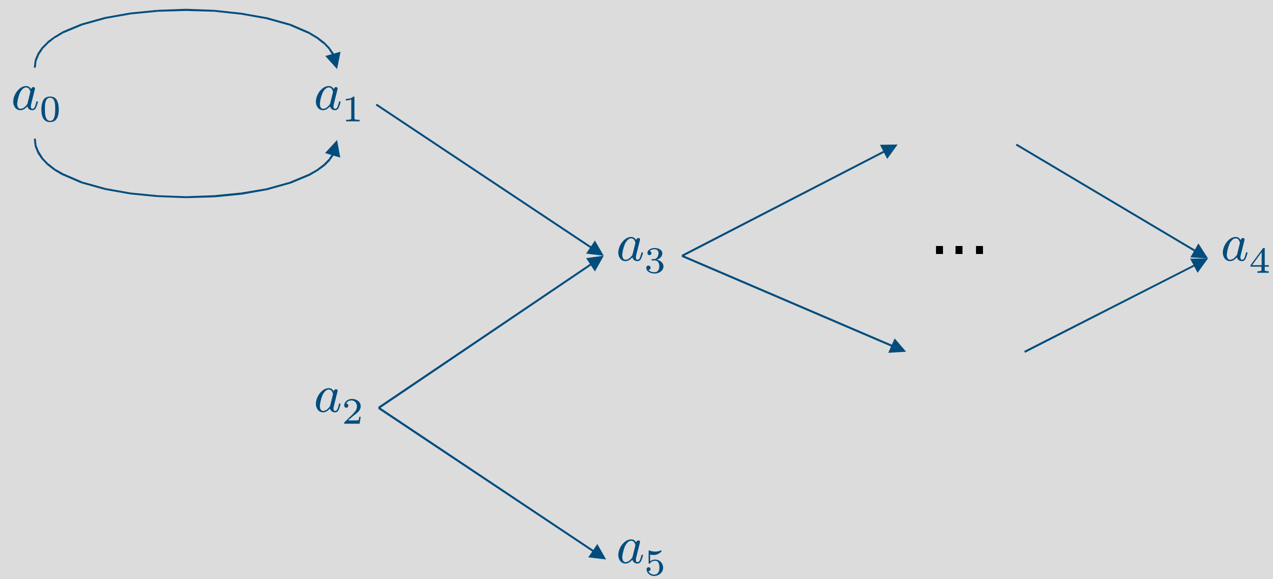
Ecole Polytechnique - France

CSL'07

14th of September, 2007

Agenda

- **Motivational Examples**
- Focusing
- Finite Successes and Finite Failures
- Applications
- Conclusions and Future Works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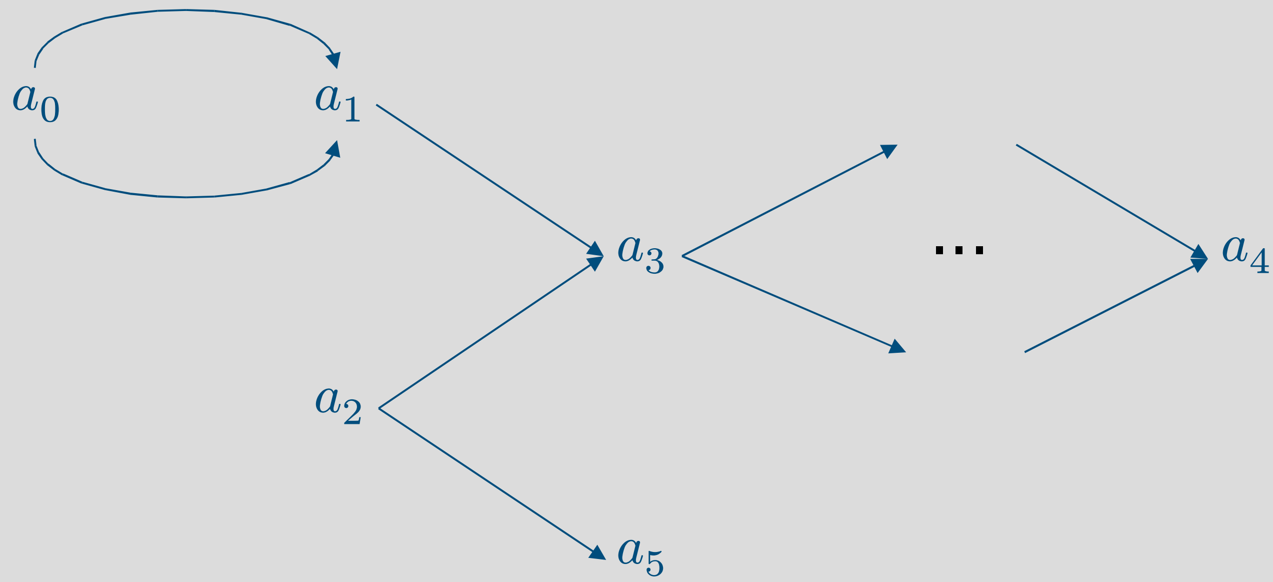
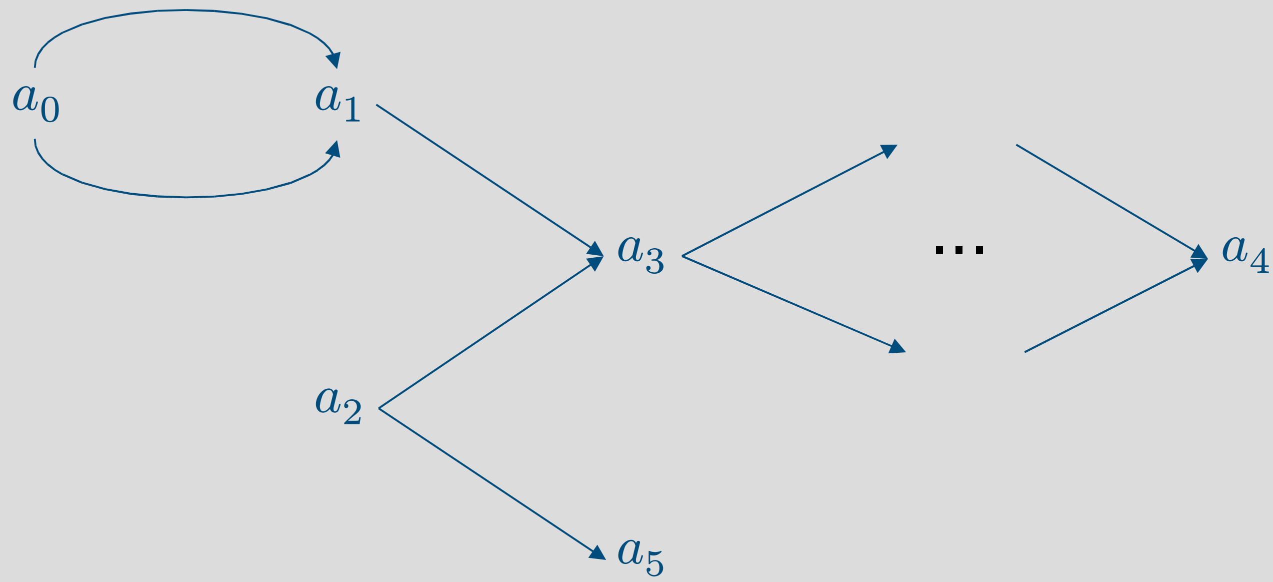


$\forall x(\mathsf{path}\ x\ x)$
$\forall x \forall y \forall z(\mathsf{arr}\ x\ z \wedge \mathsf{path}\ z\ y \supset \mathsf{path}\ x\ y)$

$$\mathsf{path}\ a_1\ a_4 \wedge \mathsf{path}\ a_2\ a_4$$



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Common Subgoal

$\text{path}\ a_3\ a_4$

- In Prolog, this common subgoal is computed twice.

How to avoid this declaratively:

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- Introduce the common subgoal with a cut

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Another example (without cuts)

- Change to an equivalent goal:

$$A \wedge G \equiv A \wedge (A \supset G)$$

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But we are only increasing non-determinism:

- There are now more proofs for the goal;
- How to give a purely **proof theoretic solution** where common subgoals aren't re-proven.

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LJF system [Liang & Miller]

$$\frac{[N,\Gamma] \xrightarrow{N} [R]}{[N,\Gamma] \longrightarrow [R]}\ D_l \qquad \frac{[\Gamma] -_{P}\!\!\rightarrow}{[\Gamma] \longrightarrow [P]}\ D_r \qquad \frac{[\Gamma], P \longrightarrow [R]}{[\Gamma] \xrightarrow{P} [R]}\ R_l \qquad \frac{[\Gamma] \longrightarrow N}{[\Gamma] -_{N}\!\!\rightarrow}\ R_r$$

$$\frac{[\Gamma, N_a], \Theta \longrightarrow \mathcal{R}}{[\Gamma], \Theta, N_a \longrightarrow \mathcal{R}}\ []_l \qquad \frac{[\Gamma], \Theta \longrightarrow [P_a]}{[\Gamma], \Theta \longrightarrow P_a}\ []_r$$

$$\frac{}{[\Gamma] \xrightarrow{A_n} [A_n]}\ I_l \qquad \frac{}{[\Gamma, A_p] -_{A_p}\!\!\rightarrow}\ I_r$$

$$\frac{}{[\Gamma], \Theta, \bot \longrightarrow \mathcal{R}}\ false_l \qquad \frac{[\Gamma], \Theta \longrightarrow \mathcal{R}}{[\Gamma], \Theta, true \longrightarrow \mathcal{R}}\ true_l \qquad \frac{}{[\Gamma] -_{true}\!\!\rightarrow}\ true_r$$

$$\frac{[\Gamma], \Theta, A, B \longrightarrow \mathcal{R}}{[\Gamma], \Theta, A \wedge B \longrightarrow \mathcal{R}}\ \wedge_l \qquad \frac{[\Gamma] -_{A}\!\!\rightarrow \quad [\Gamma] -_{B}\!\!\rightarrow}{[\Gamma] -_{A \wedge B}\!\!\rightarrow}\ \wedge_r \qquad \frac{[\Gamma] -_{A}\!\!\rightarrow \quad [\Gamma] \xrightarrow{B} [R]}{[\Gamma] \xrightarrow{A \supset B} [R]}\ \supset_l \qquad \frac{[\Gamma] -_{A}\!\!\rightarrow \quad [\Gamma] \xrightarrow{B} [R]}{[\Gamma] \xrightarrow{A \supset B} [R]}\ \supset_l$$

$$\frac{[\Gamma], \Theta, A \longrightarrow \mathcal{R}}{[\Gamma], \Theta, \exists y A \longrightarrow \mathcal{R}}\ \exists_l \qquad \frac{[\Gamma] -_{A[t/x]}\!\!\rightarrow}{[\Gamma] -_{\exists x A}\!\!\rightarrow}\ \exists_r \qquad \frac{[\Gamma] \xrightarrow{A[t/x]} [R]}{[\Gamma] \xrightarrow{\forall x A} [R]}\ \forall_l \qquad \frac{[\Gamma], \Theta \longrightarrow A}{[\Gamma], \Theta \longrightarrow \forall y A}\ \forall_r$$

Playing with polarities

$\{a, a \supset b, b \supset c\}$

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b is negative

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  \dfrac{
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      \dfrac{
        \dfrac{
          \dfrac{}{[\Gamma] -_a\!\!\longrightarrow}\, I_r
          \qquad
          \dfrac{}{[\Gamma] \xrightarrow{\;b\;} [b]}\, I_l
        }{[\Gamma] \xrightarrow{a \supset b} [b]}\, \supset_l
      }{[\Gamma] \longrightarrow [b]}\, D_l
    }{[\Gamma] \longrightarrow b}\, []_r
  }{[\Gamma] -_b\!\!\longrightarrow}\, R_r
  \qquad
  \dfrac{}{[\Gamma] \xrightarrow{\;c\;} [c]}\, I_l
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$$

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b is positive:

$$\dfrac{\dfrac{}{[\Gamma] -_a\!\rightarrow}\,I_r \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{[\Gamma] -_b\!\rightarrow}\,I_r \quad \dfrac{}{[\Gamma] \xrightarrow{c} [c]}\,I_l}{[\Gamma, b] \xrightarrow{b \supset c} [c]}\,\supset_l}{[\Gamma, b] \longrightarrow [c]}\,D_l}{[\Gamma], b \longrightarrow [c]}\,[]_l}{[\Gamma] \xrightarrow{b} [c]}\,R_l}{[\Gamma] \xrightarrow{a \supset b} [c]}\,\supset_l$$

Changing polarities doesn't affect provability:

LJF^t

$$\mathcal{P}; \Gamma \longrightarrow G$$

$$\frac{\mathcal{P}; [\Gamma] \longrightarrow L_1 \qquad \cdots \qquad \mathcal{P}; [\Gamma] \longrightarrow L_n \qquad \mathcal{P} \cup \Delta_P; [\Gamma \cup \Delta_L] \longrightarrow [R]}{\mathcal{P}; [\Gamma] \longrightarrow [R]}\ mc.$$

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Proposition: LJF^t is sound and complete w.r.t LJF

- The idea is to assign at the base of the tree negative polarity to all atoms, and then use the mulitcut rule to change the polarity of some atoms to positive polarity.

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A **Table** is a partially ordered set of lemmas, and it specifies a multicut derivation

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$$\mathcal{T} = \langle \mathcal{A}, \prec \rangle$$

minimal elements

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Multicut Derivation

$$\frac{\Gamma \longrightarrow A_1 \quad \cdots \quad \Gamma \longrightarrow A_n \qquad \Gamma, A_1, \ldots, A_n \longrightarrow G}{\Gamma \longrightarrow G}$$

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Where do tables come from:

- Lemmas in an Interactive Theorem Prover – Boyer & Moore;
- Logic Programming – *Tabled Deduction*;
- Extracted from an known proof (e.g. depth first traversal) - *Proof Carrying Code*;

Consider: Horn Theory and Tables composed only of atoms

Proposition: Let Γ be a set of Horn clauses, $A \in \mathcal{P} \cap \Gamma$, and Ξ be an arbitrary LJF^t proof tree for $\mathcal{P}; [\Gamma] -_G \rightarrow$. Then every occurrence of a sequent with right-hand side the atom A is the conclusion of an I_r^t rule.

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}{\mathsf{path}\ a_3\ a_4; [\Gamma, \mathsf{path}\ a_3\ a_4] -_{\mathsf{arr}_{a_1\ a_3}}\rightarrow}\ R_r^t
\quad
\dfrac{}{\mathsf{path}\ a_3\ a_4; [\Gamma, \mathsf{path}\ a_3\ a_4] -_{\mathsf{path}\ a_3\ a_4}\rightarrow}\ I_r^t
}{
\dfrac{\mathsf{path}\ a_3\ a_4; [\Gamma, \mathsf{path}\ a_3\ a_4] -_{\mathsf{arr}_{a_1\ a_3} \wedge \mathsf{path}\ a_3\ a_4}\rightarrow}{\mathsf{path}\ a_3\ a_4; [\Gamma, \mathsf{path}\ a_3\ a_4] \longrightarrow [\mathsf{path}\ a_1\ a_4]}\ D_l^t, \forall_l^t, \forall_l^t, \forall_l^t, \supset_l^t
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Returning to the Examples

$$\frac{\Gamma \longrightarrow A \qquad \Gamma \longrightarrow G}{\Gamma \longrightarrow A \wedge G} \quad \Longrightarrow \quad \frac{\mathcal{P}; [\Gamma] \longrightarrow [A] \qquad \mathcal{P} \cup \{A\}; [\Gamma, A] \longrightarrow [A \wedge G]}{\mathcal{P}; [\Gamma] \longrightarrow [A \wedge G]}\ mc.$$

Consider: Horn Theory and Tables composed by literals

A well studied way to handle **negation-as-finite-failure** in sequent calculus is with the use of definitions (fixed points):

$$\frac{\{\mathcal{P}; [\Gamma\theta], \Theta\theta, B\theta \longrightarrow \mathcal{R}\theta \mid \theta = mgu(H, A) \text{ for some clause } H \triangleq B\}}{\mathcal{P}; [\Gamma], \Theta, A \longrightarrow \mathcal{R}} \;\mathsf{Def}_l,\, A \notin \mathcal{P}$$

$$\frac{\mathcal{P}; [\Gamma] -_{B\theta}\!\!\rightarrow}{\mathcal{P}; [\Gamma] -_{A_n}\!\!\rightarrow} \;\mathsf{Def}_r,\, A_n \notin \mathcal{P},\ \text{where } H \triangleq B, \text{ and } H\theta = A_n$$

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Proposition: Let Γ be a set of literals built on positive polarity atoms, and let $L \in \Gamma$. If Ξ is proof of $\mathcal{P}; [\Gamma] -_G\!\!\rightarrow$ then all occurrences of sequents in Ξ that have L as their right-focus formula are the conclusion of a proof with Def_r-depth at most 1.

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Declarative Interpretation for *Tabled Deduction*

Both finite successes and finite failures are stored in a table, and if:

- There is another attempt to prove a **finite success**, the proof search ends with success;
  - **Follows from the previous result;**

Declarative Interpretation for *Tabled Deduction*

Both finite successes and finite failures are stored in a table, and if:

- There is another attempt to prove a **finite success**, the proof search ends with success;
  - **Follows from the previous result;**
- There is another attempt to prove a **finite failure**, this proof search fails immediately
  - **Follows from the following result:**

Proposition: Let A be an atom such that $\Gamma \longrightarrow A$ is not provable and let $A \in \mathcal{P}$. Let Ξ be an arbitrary $LJF^{\Delta t}$ derivation for $\mathcal{P}; [\Gamma] -_G \rightarrow$. Then all sequents in Ξ with right-hand side A are open leaves.

Proof Carrying Code – Tables as Proof Objects

If tables are used as proof objects they seem to enjoy the following crucial properties:

- Small;
- Easy to Check;
- Flexible.

Proposition: Let Ξ be a LJF proof of $\Gamma \longrightarrow G$ and let $\mathcal{T}$ be a table obtained from Ξ using a postorder traversal. There exists a proof for $mcd(\mathcal{T}, [.]\Gamma \longrightarrow G)$ such that all of its added subproofs have decide-depth of one or less.

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Conclusions and Future Works

- **Extend these results to stronger logics**
  - Hereditary Harrop Formulas;
  - mu-Mall (Sequent calculus with Induction and Co-induction) [Baelde & Miller – LPAR'07];
  - Experiments.
- **Investigate connections with Interactive Theorem Proving**
  - Use a sequence of lemmas to prove a theorem in such a way that the gaps between them are “easy” to be found.