Focusing in Linear Meta Logic

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Agenda

- **Overview**
- Linear Logic and Focusing
- Encoding Systems
- Results
- Conclusions and Future Works

Overview

One specification $\mathcal{L}$

Some logical equivalences $F \equiv F'$

Several Systems

Sequent Calculus

Natural Deduction

Tableaux Systems

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Linear Logic - Basics

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de Morgan dualities

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$? / !$ $\quad 1/0$

$A \multimap B$ denotes $B ⅋ A^{\bot}$

$\neg B \Rightarrow$ Negation normal form

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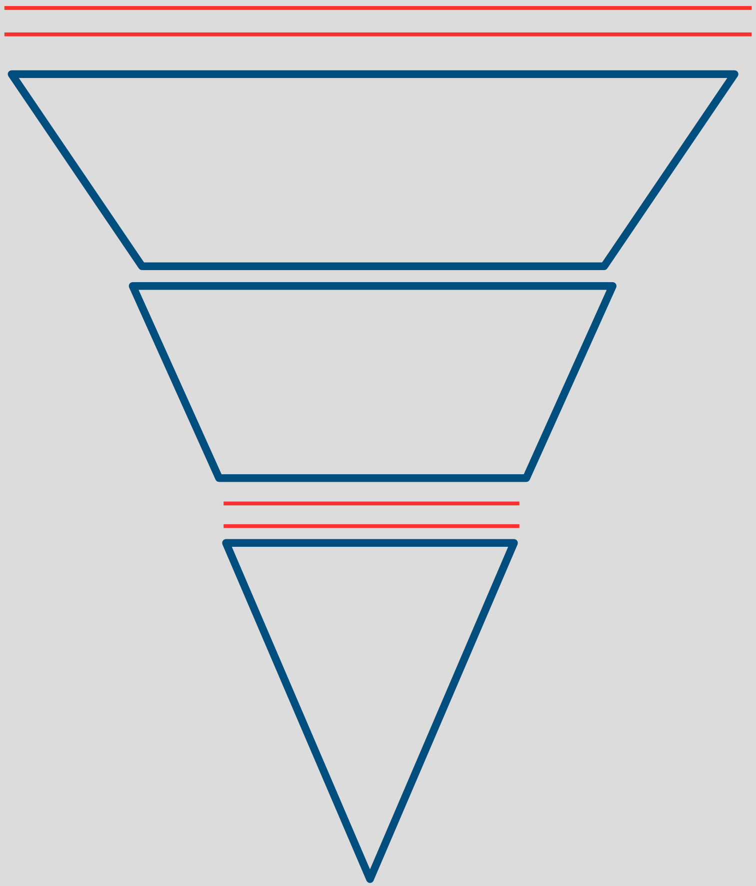
$A \multimap B$ denotes $B \parr A^{\perp}$

$\neg B$ ⟹ Negation normal form

$$B \multimap C, \Delta \vdash \Gamma \quad \Longrightarrow \quad \vdash C^{\perp} \otimes B, \Delta^{\perp}, \Gamma$$

Focusing

Focusing



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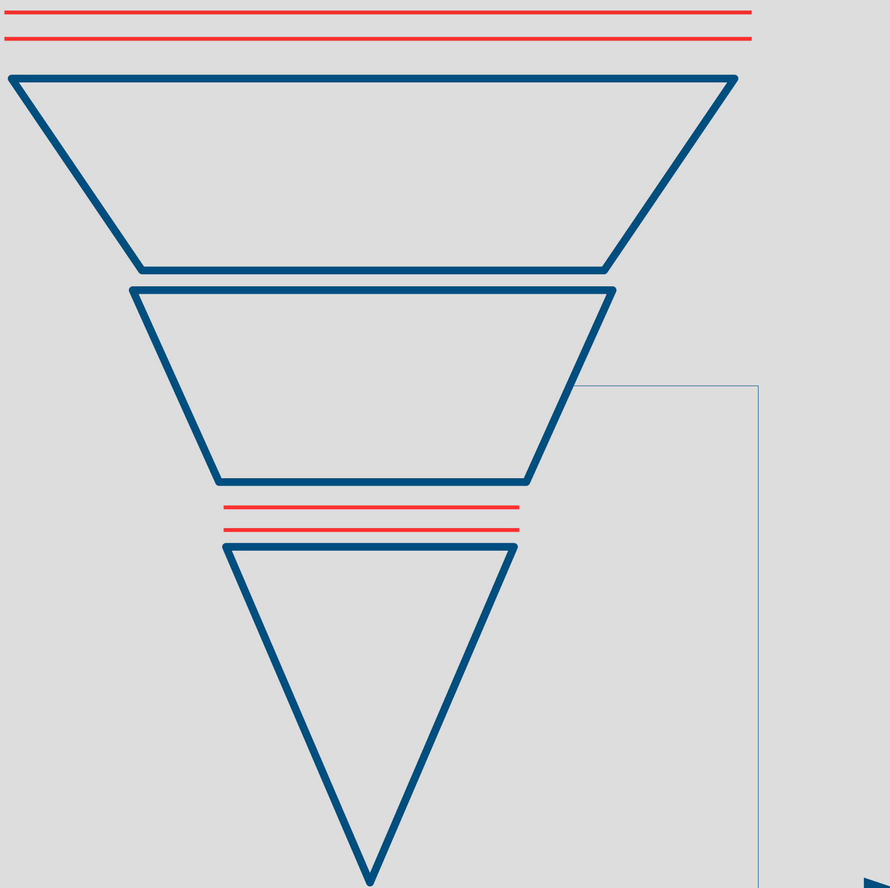

Negative Phase - All invertible rules are applied eagerly

$$\frac{\vdash \Theta : \Gamma \Uparrow L, F, G}{\vdash \Theta : \Gamma \Uparrow L, F \parr G}\ [\parr]$$

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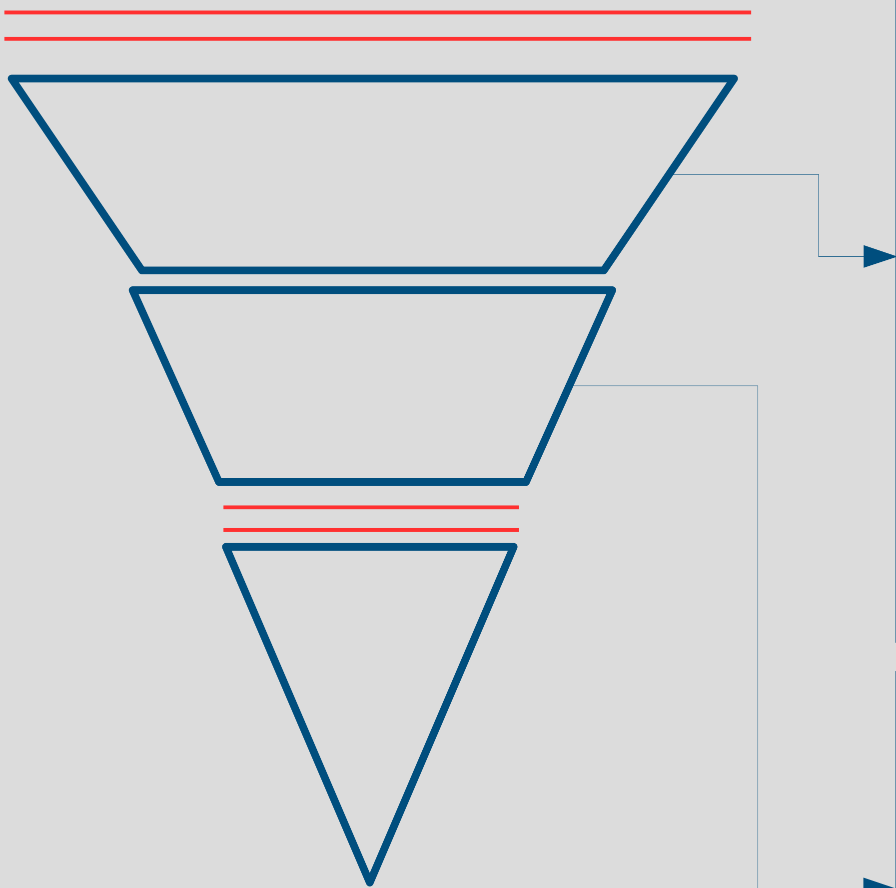

Positive Phase – One formula is focused on

$$\frac{\vdash \Theta : \Gamma \Downarrow P}{\vdash \Theta : \Gamma, P \Uparrow} \; [D_1]$$

Focusing persists

$$\frac{\vdash \Theta : \Gamma \Downarrow F \quad \vdash \Theta : \Gamma' \Downarrow G}{\vdash \Theta : \Gamma, \Gamma' \Downarrow F \otimes G} \; [\otimes]$$

Negative Phase - All invertible rules are applied eagerly

$$\frac{\vdash \Theta : \Gamma \Uparrow L, F, G}{\vdash \Theta : \Gamma \Uparrow L, F \parr G} \; [\parr]$$

Focusing Basics

$A \mathbin{\&} B, A \parr B, \bot, \top, ?B, \forall x\, B$ Negative Formulas

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$$\frac{\vdash \Theta : \Gamma \Uparrow L}{\vdash \Theta : \Gamma \Uparrow L, \bot}\ [\bot] \qquad \frac{\vdash \Theta : \Gamma \Uparrow L, F, G}{\vdash \Theta : \Gamma \Uparrow L, F \parr G}\ [\parr] \qquad \frac{\vdash \Theta, F : \Gamma \Uparrow L}{\vdash \Theta : \Gamma \Uparrow L, ?F}\ [?]$$

$$\frac{}{\vdash \Theta : \Gamma \Uparrow L, \top}\ [\top] \qquad \frac{\vdash \Theta : \Gamma \Uparrow L, F \quad \vdash \Theta : \Gamma \Uparrow L, G}{\vdash \Theta : \Gamma \Uparrow L, F \mathbin{\&} G}\ [\&] \qquad \frac{\vdash \Theta : \Gamma \Uparrow L, F[c/x]}{\vdash \Theta : \Gamma \Uparrow L, \forall x\, F}\ [\forall]$$

All negative rules are invertible

Focusing Basics

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$$A \otimes B, A \oplus B, 1, !\, B, \exists x\, B$$ Positive Formulas

$$\frac{}{\vdash \Theta :\Downarrow 1}\ [1] \qquad \frac{\vdash \Theta : \Gamma \Downarrow F \quad \vdash \Theta : \Gamma' \Downarrow G}{\vdash \Theta : \Gamma, \Gamma' \Downarrow F \otimes G}\ [\otimes] \qquad \frac{\vdash \Theta :\Uparrow F}{\vdash \Theta :\Downarrow\, !\, F}\ [!]$$

$$\frac{\vdash \Theta : \Gamma \Downarrow F}{\vdash \Theta : \Gamma \Downarrow F \oplus G}\ [\oplus_l] \qquad \frac{\vdash \Theta : \Gamma \Downarrow G}{\vdash \Theta : \Gamma \Downarrow F \oplus G}\ [\oplus_r] \qquad \frac{\vdash \Theta, F : \Gamma \Downarrow F[t/x]}{\vdash \Theta : \Gamma \Downarrow \exists x\, F}\ [\exists]$$

$$\frac{\vdash \Theta : \Gamma \Downarrow P}{\vdash \Theta : \Gamma, P \Uparrow}\ [D_1] \qquad \frac{\vdash \Theta, P : \Gamma \Downarrow P}{\vdash \Theta, P : \Gamma \Uparrow}\ [D_2] \qquad \frac{\vdash \Theta : \Gamma \Uparrow N}{\vdash \Theta : \Gamma \Downarrow N}\ [R \Downarrow] \qquad \frac{\vdash \Theta : \Gamma, S \Uparrow L}{\vdash \Theta : \Gamma \Uparrow L, S}\ [R \Uparrow]$$

Positive rules are not necessarily invertible.

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$$\frac{}{\vdash \Theta : A_p^{\perp} \Downarrow A_p}\ [I_1] \qquad \frac{}{\vdash \Theta, A_p^{\perp} :\Downarrow A_p}\ [I_2]$$

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The Focusing Theorem states that a formula is provable in the focused system iff it is provable in linear logic. Does not matter how we assign the polarity of literals.

Playing with polarities

Fibonacci Program

$$\mathrm{fib}(0,0) \wedge \mathrm{fib}(1,1) \wedge$$
$$\forall n, f, f' [\mathrm{fib}(n, f) \supset \mathrm{fib}(n+1, f') \supset \mathrm{fib}(n+2, f+f')].$$

To prove

$$\Gamma \longrightarrow \mathrm{fib}(n, f_n).$$

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there are infinitely many proofs and the smallest one is of linear size in ***n*** (program-directed, forward-chaining).

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there are infinitely many proofs and the smallest one is of linear size in ***n*** (program-directed, forward-chaining).

While choices in the polarization of atoms **do not affect provability**, it can have important consequences on the **shape of proofs**.

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Encoding Formulas

OL

- Sequent Calculus – Left / Right
- Natural Deduction – Hyp / Con
- Tableaux – Neg / Pos

ML

$\lfloor \cdot \rfloor$ $\lceil \cdot \rceil$ $\mathrm{form} \to o$

Encoding Logics

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	Encoding Formulas	Encoding Sequents
OL	• Sequent Calculus – Left / Right • Natural Deduction – Hyp / Con • Tableaux – Neg / Pos	$B_1, \ldots, B_n \vdash C_1, \ldots, C_m$
ML	$\lfloor \cdot \rfloor \quad \lceil \cdot \rceil \quad \text{form} \rightarrow o$	$\vdash \lfloor B_1 \rfloor, \ldots, \lfloor B_n \rfloor, \lceil C_1 \rceil, \ldots, \lceil C_m \rceil$

Theory $\mathcal{L}$ with the meaning of connectives – Existential Closure of

$(\Rightarrow_L)$	$\lfloor A \Rightarrow B \rfloor^{\perp} \otimes (\lceil A \rceil \otimes \lfloor B \rfloor)$	$(\Rightarrow_R)$	$\lceil A \Rightarrow B \rceil^{\perp} \otimes (\lfloor A \rfloor \parr \lceil B \rceil)$
$(\wedge_L)$	$\lfloor A \wedge B \rfloor^{\perp} \otimes (\lfloor A \rfloor \oplus \lfloor B \rfloor)$	$(\wedge_R)$	$\lceil A \wedge B \rceil^{\perp} \otimes (\lceil A \rceil \mathbin{\&} \lceil B \rceil)$
$(\forall_L)$	$\lfloor \forall B \rfloor^{\perp} \otimes \lfloor Bx \rfloor$	$(\forall_R)$	$\lceil \forall B \rceil^{\perp} \otimes \forall x \lceil Bx \rceil$
$(\perp_L)$	$\lfloor \perp \rfloor^{\perp}$	(t_R)	$\lceil t \rceil^{\perp} \otimes \top$

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$(\wedge_L)$	$\lfloor A \wedge B \rfloor^{\perp} \otimes (\lfloor A \rfloor \oplus \lfloor B \rfloor)$	$(\wedge_R)$	$\lceil A \wedge B \rceil^{\perp} \otimes (\lceil A \rceil \,\&\, \lceil B \rceil)$
$(\forall_L)$	$\lfloor \forall B \rfloor^{\perp} \otimes \lfloor Bx \rfloor$	$(\forall_R)$	$\lceil \forall B \rceil^{\perp} \otimes \forall x \lceil Bx \rceil$
$(\perp_L)$	$\lfloor \perp \rfloor^{\perp}$	(t_R)	$\lceil t \rceil^{\perp} \otimes \top$

and the structural and identity rules

(Id_1)	$\lfloor B \rfloor^{\perp} \otimes \lceil B \rceil^{\perp}$	(Id_2)	$\lfloor B \rfloor \otimes \lceil B \rceil$
(Str_L)	$\lfloor B \rfloor^{\perp} \otimes ?\lfloor B \rfloor$	(Str_R)	$\lceil B \rceil^{\perp} \otimes ?\lceil B \rceil$
(W_R)	$\lceil C \rceil^{\perp} \otimes \perp$		

Duality of the $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ atoms

$$\vdash \forall B(\lceil B \rceil \equiv \lfloor B \rfloor^{\perp}) \,\&\, \forall B(\lfloor B \rfloor \equiv \lceil B \rceil^{\perp}), \mathsf{Id}_1, \mathsf{Id}_2$$

with Str_L and Str_R we prove the equivalences:

$$\lfloor B \rfloor \equiv ?\lfloor B \rfloor \text{ and } \lceil B \rceil \equiv ?\lceil B \rceil$$

Levels of Adequacy

We identify three levels of adequacy:

- Relative completeness: comparisons deal only with provability: the two systems have the same theorems.
- Full completeness of proofs: comparisons deal with proof objects: the proofs of a given formula are in one-to-one correspondence with proofs in another system.
- Full completeness of derivations: comparisons deal with derivations (*i.e.*, open proofs, such as inference rules themselves): the derivations in one system are in one-to-one correspondence with those in another system.

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- Full completeness of derivations: comparisons deal with derivations (*i.e.*, open proofs, such as inference rules themselves): the derivations in one system are in one-to-one correspondence with those in another system.

We try for the adequacy on the **level of derivations**, but sometimes we settle for the **level of proofs**.

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Sequent Calculus

Sequent Calculus

if all $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ (meta-level) atoms are negative

1) $\Gamma \vdash_{lm} C$ iff $\vdash \mathcal{L}_{lm}, \lfloor\Gamma\rfloor : \lceil C\rceil \Uparrow$
2) $\Gamma \vdash_{lj} C$ iff $\vdash \mathcal{L}_{lj}, \lfloor\Gamma\rfloor : \lceil C\rceil \Uparrow$
3) $\Gamma \vdash_{lk} \Delta$ iff $\vdash \mathcal{L}_{lk}, \lfloor\Gamma\rfloor, \lceil\Delta\rceil :\Uparrow$

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$\mathcal{L}_{lk} = \mathcal{L} \cup \{\mathsf{Id}_1, \mathsf{Id}_2, \mathsf{Str}_L, \mathsf{Str}_R\}$,
$\mathcal{L}_{lm} = \mathcal{L} \cup \{\mathsf{Id}_1, \mathsf{Id}_2, \mathsf{Str}_L, \Rightarrow'_L\} \setminus \{\bot_L, \Rightarrow_L\}$,
$\mathcal{L}_{lj} = \mathcal{L} \cup \{\mathsf{Id}_1, \mathsf{Id}_2, \mathsf{Str}_L, \Rightarrow'_L, W_R\} \setminus \{\Rightarrow_L\}$, and
$\Rightarrow'_L$ is $?\exists A \exists B[\lfloor A \Rightarrow B \rfloor^\perp \otimes (!\lceil A \rceil \otimes \lfloor B \rfloor)]$

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$$\mathcal{L}_{lk} = \mathcal{L} \cup \{\mathsf{Id}_1, \mathsf{Id}_2, \mathsf{Str}_L, \mathsf{Str}_R\},$$
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$$\mathcal{L}_{lj} = \mathcal{L} \cup \{\mathsf{Id}_1, \mathsf{Id}_2, \mathsf{Str}_L, \Rightarrow'_L, W_R\} \setminus \{\Rightarrow_L\}, \text{ and}$$
$$\Rightarrow'_L \text{ is } ?\exists A \exists B[\lfloor A \Rightarrow B\rfloor^{\perp} \otimes (!\lceil A\rceil \otimes \lfloor B\rfloor)]$$

We can also obtain a adequacy up to the level of derivations. For intuitionistic and minimal logics the ! is important.

The encoding of an inference rule (remember all meta-level atoms are negative):

$$\frac{\Gamma \vdash A \qquad \Gamma, B \vdash C}{\Gamma, A \Rightarrow B \vdash C}$$

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$$\dfrac{\dfrac{}{\vdash \mathcal{K} :\Downarrow \lfloor A \Rightarrow B \rfloor^{\perp}}\,[I_2] \qquad \dfrac{\dfrac{\vdash \mathcal{K} : \lceil A \rceil \Uparrow}{\vdash \mathcal{K} :\Downarrow !\lceil A \rceil}\,[!, R\Uparrow] \qquad \dfrac{\vdash \mathcal{K} : \lfloor B \rfloor, \lceil C \rceil \Uparrow}{\vdash \mathcal{K} : \lceil C \rceil \Downarrow \lfloor B \rfloor}\,[R\Downarrow, R\Uparrow]}{\vdash \mathcal{K} : \lceil C \rceil \Downarrow !\lceil A \rceil \otimes \lfloor B \rfloor}\,[\otimes]}{\dfrac{\vdash \mathcal{K} : \lceil C \rceil \Downarrow F}{\vdash \mathcal{K} : \lceil C \rceil \Uparrow \cdot}\,[D_2]}\,[2 \times \exists, \otimes]$$

F is $\exists A \exists B \lfloor A \Rightarrow B \rfloor^{\perp} \otimes (\lceil A \rceil \otimes \lfloor B \rfloor)$

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$$\dfrac{\Gamma \vdash A \qquad\qquad \Gamma, B \vdash C}{\Gamma, A \Rightarrow B \vdash C}$$

$\lfloor A \Rightarrow B \rfloor \in \mathcal{K}$
is enforced

$$\cfrac{\cfrac{\cfrac{}{\vdash \mathcal{K} :\Downarrow \lfloor A \Rightarrow B \rfloor^{\perp}}\;[I_2] \qquad \cfrac{\cfrac{\vdash \mathcal{K} : \lceil A \rceil \Uparrow}{\vdash \mathcal{K} :\Downarrow !\lceil A \rceil}\;[!, R\Uparrow] \qquad \cfrac{\vdash \mathcal{K} : \lfloor B \rfloor, \lceil C \rceil \Uparrow}{\vdash \mathcal{K} : \lceil C \rceil \Downarrow \lfloor B \rfloor}\;[R\Downarrow, R\Uparrow]}{\vdash \mathcal{K} : \lceil C \rceil \Downarrow !\lceil A \rceil \otimes \lfloor B \rfloor}\;[\otimes]}{\vdash \mathcal{K} : \lceil C \rceil \Downarrow F}\;[2 \times \exists, \otimes]}{\vdash \mathcal{K} : \lceil C \rceil \Uparrow \cdot}\;[D_2]$$

F is $\exists A \exists B \lfloor A \Rightarrow B \rfloor^{\perp} \otimes (\lceil A \rceil \otimes \lfloor B \rfloor)$

The encoding of an inference rule (remember all meta-level atoms are negative):

$$\frac{\Gamma \vdash A \qquad \Gamma, B \vdash C}{\Gamma, A \Rightarrow B \vdash C}$$

$\lceil C \rceil$ must go to the right branch

$\lfloor A \Rightarrow B \rfloor \in \mathcal{K}$ is enforced

$$\dfrac{\dfrac{}{\vdash \mathcal{K} :\Downarrow \lfloor A \Rightarrow B \rfloor^{\perp}}\,[I_2] \qquad \dfrac{\dfrac{\vdash \mathcal{K} : \lceil A \rceil \Uparrow}{\vdash \mathcal{K} :\Downarrow !\lceil A \rceil}\,[!, R\Uparrow] \qquad \dfrac{\vdash \mathcal{K} : \lfloor B \rfloor, \lceil C \rceil \Uparrow}{\vdash \mathcal{K} : \lceil C \rceil \Downarrow \lfloor B \rfloor}\,[R\Downarrow, R\Uparrow]}{\vdash \mathcal{K} : \lceil C \rceil \Downarrow !\lceil A \rceil \otimes \lfloor B \rfloor}\,[\otimes]}{\dfrac{\vdash \mathcal{K} : \lceil C \rceil \Downarrow F}{\vdash \mathcal{K} : \lceil C \rceil \Uparrow \cdot}\,[D_2]}\,[2 \times \exists, \otimes]$$

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Cut free proofs – remove the clause (ID_2) from the theory:

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It is possible to obtain an adequacy on the level of derivations.

Natural Deduction [Sieg, Byrnes, 1998]

$$\frac{}{\Gamma, A \vdash_{nd} A \downarrow}\ [\mathrm{Ax}] \qquad \frac{\Gamma \vdash_{nd} F \uparrow \quad \Gamma \vdash_{nd} G \uparrow}{\Gamma \vdash_{nd} F \wedge G \uparrow}\ [\wedge I] \qquad \frac{\Gamma \vdash_{nd} F \wedge G \downarrow}{\Gamma \vdash_{nd} F \downarrow}\ [\wedge E]$$

$$\frac{\Gamma, A \vdash_{nd} B \uparrow}{\Gamma \vdash_{nd} A \Rightarrow B \uparrow}\ [\Rightarrow I] \qquad \frac{\Gamma \vdash_{nd} A \Rightarrow B \downarrow \quad \Gamma \vdash_{nd} A \uparrow}{\Gamma \vdash_{nd} B \downarrow}\ [\Rightarrow E] \qquad \frac{}{\Gamma \vdash_{nd} t \uparrow}\ [tI]$$

$$\frac{\Gamma \vdash_{nd} A\{c/x\} \uparrow}{\Gamma \vdash_{nd} \forall x\, A \uparrow}\ [\forall I] \qquad \frac{\Gamma \vdash_{nd} \forall x\, A \downarrow}{\Gamma \vdash_{nd} A\{t/x\} \downarrow}\ [\forall E] \qquad \frac{\Gamma \vdash_{nd} A \downarrow}{\Gamma \vdash_{nd} A \uparrow}\ [\mathrm{M}] \qquad \frac{\Gamma \vdash_{nd} A \uparrow}{\Gamma \vdash_{nd} A \downarrow}\ [\mathrm{S}]$$

Natural Deduction [Sieg, Byrnes, 1998]

$$\frac{}{\Gamma, A \vdash_{nd} A \downarrow}\,[\mathrm{Ax}] \quad \frac{\Gamma \vdash_{nd} F \uparrow \quad \Gamma \vdash_{nd} G \uparrow}{\Gamma \vdash_{nd} F \wedge G \uparrow}\,[\wedge I] \quad \frac{\Gamma \vdash_{nd} F \wedge G \downarrow}{\Gamma \vdash_{nd} F \downarrow}\,[\wedge E]$$

$$\frac{\Gamma, A \vdash_{nd} B \uparrow}{\Gamma \vdash_{nd} A \Rightarrow B \uparrow}\,[\Rightarrow I] \quad \frac{\Gamma \vdash_{nd} A \Rightarrow B \downarrow \quad \Gamma \vdash_{nd} A \uparrow}{\Gamma \vdash_{nd} B \downarrow}\,[\Rightarrow E] \quad \frac{}{\Gamma \vdash_{nd} t \uparrow}\,[tI]$$

$$\frac{\Gamma \vdash_{nd} A\{c/x\} \uparrow}{\Gamma \vdash_{nd} \forall x\, A \uparrow}\,[\forall I] \quad \frac{\Gamma \vdash_{nd} \forall x\, A \downarrow}{\Gamma \vdash_{nd} A\{t/x\} \downarrow}\,[\forall E] \quad \frac{\Gamma \vdash_{nd} A \downarrow}{\Gamma \vdash_{nd} A \uparrow}\,[\mathrm{M}] \quad \frac{\Gamma \vdash_{nd} A \uparrow}{\Gamma \vdash_{nd} A \downarrow}\,[\mathrm{S}]$$

$$\Gamma \vdash_{nd} C \uparrow \qquad \vdash \Sigma, \lfloor \Gamma \rfloor : \lceil C \rceil \Uparrow$$

Useful to identify normal proofs, where the S rules is not allowed.

$$\Gamma \vdash_{nd} C \downarrow \qquad \vdash \Sigma, \lfloor \Gamma \rfloor : \lfloor C \rfloor^{\perp} \Uparrow$$

Natural Deduction – including normal forms

if all $\lceil\cdot\rceil$ (meta-level) atoms are negative

if all $\lfloor\cdot\rfloor$ (meta-level) atoms are positive

1) $\Gamma \vdash_{nm} C\!\uparrow$ iff $\vdash \mathcal{L}_{lm}, \lfloor\Gamma\rfloor : \lceil C\rceil \Uparrow$

2) $\Gamma \vdash^{n}_{nm} C\!\uparrow$ iff $\vdash \mathcal{L}^{f}_{lm}, \lfloor\Gamma\rfloor : \lceil C\rceil \Uparrow$

3) $\Gamma \vdash^{n}_{nm} C\!\downarrow$ iff $\vdash \mathcal{L}^{f}_{lm}, \lfloor\Gamma\rfloor : \lfloor C\rfloor^{\perp} \Uparrow$

An adequacy on the level of proofs can also be obtained.

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An adequacy on the level of proofs can also be obtained.

Since the polarity assignment a focused system does not affect provability, we obtain the following relative completeness result for free:

Corollary

$$\Gamma \vdash_{lm} C \text{ iff } \Gamma \vdash_{nm} C \quad \text{and} \quad \Gamma \vdash^{f}_{lm} C \text{ iff } \Gamma \vdash^{n}_{nm} C.$$

Cut now becomes Switch Rule:

$$\frac{\Gamma \vdash_{nd} C \uparrow}{\Gamma \vdash_{nd} C \downarrow} \text{[S]}$$

$$\dfrac{\dfrac{}{\vdash \Sigma, \lfloor \Gamma \rfloor : \lfloor C \rfloor^{\perp} \Downarrow \lfloor C \rfloor} [I_1] \quad \dfrac{\vdash \Sigma, \lfloor \Gamma \rfloor : \lceil C \rceil \Uparrow}{\vdash \Sigma, \lfloor \Gamma \rfloor :\Downarrow \lceil C \rceil} [R \Downarrow, R \Uparrow]}{\dfrac{\vdash \Sigma, \lfloor \Gamma \rfloor : \lfloor C \rfloor^{\perp} \Downarrow \lfloor C \rfloor \otimes \lceil C \rceil}{\vdash \Sigma, \lfloor \Gamma \rfloor : \lfloor C \rfloor^{\perp} \Uparrow} [D_2, \exists]} [\otimes]$$

We skip the natural deduction treatment of negation in intuitionistic and classical logics.

Natural Deduction with Generalized Elimination Rules

$$\frac{\Gamma \vdash_{ge} A \vee B \quad \Gamma, A \vdash_{ge} C \quad \Gamma, B \vdash_{ge} C}{\Gamma \vdash_{ge} C} \qquad \frac{\Gamma \vdash_{ge} A \wedge B \quad \Gamma, A, B \vdash_{ge} C}{\Gamma \vdash_{ge} C}$$

$$\frac{\Gamma \vdash_{ge} A \Rightarrow B \quad \Gamma \vdash_{ge} A \quad \Gamma, B \vdash_{ge} C}{\Gamma \vdash_{ge} C} \qquad \frac{\Gamma \vdash_{ge} \forall x\, A \quad \Gamma, A\{t/x\} \vdash_{ge} C}{\Gamma \vdash_{ge} C}$$

Natural Deduction with Generalized Elimination Rules

$$\frac{\Gamma \vdash_{ge} A \vee B \quad \Gamma, A \vdash_{ge} C \quad \Gamma, B \vdash_{ge} C}{\Gamma \vdash_{ge} C} \qquad \frac{\Gamma \vdash_{ge} A \wedge B \quad \Gamma, A, B \vdash_{ge} C}{\Gamma \vdash_{ge} C}$$

$$\frac{\Gamma \vdash_{ge} A \Rightarrow B \quad \Gamma \vdash_{ge} A \quad \Gamma, B \vdash_{ge} C}{\Gamma \vdash_{ge} C} \qquad \frac{\Gamma \vdash_{ge} \forall x\, A \quad \Gamma, A\{t/x\} \vdash_{ge} C}{\Gamma \vdash_{ge} C}$$

We use the identity and structural equivalences:

$\lfloor B \rfloor^{\perp} \to \lceil B \rceil$ change $\oplus$ to $\parr$

An adequacy on the level of proofs can also be obtained.

Corollary. $\Gamma \vdash_{ge} C$ iff $\Gamma \vdash_{lm} C$.

Free Deduction

$$\frac{\Gamma, A \vee B \vdash_{fd} \Delta \quad \Gamma \vdash_{fd} \Delta, A}{\Gamma \vdash_{fd} \Delta} [\vee GI] \qquad \frac{\Gamma, A \Rightarrow B \vdash_{fd} \Delta \quad \Gamma, A \vdash_{fd} \Delta, B}{\Gamma \vdash_{fd} \Delta} [\Rightarrow GI]$$

$$\frac{\Gamma, A \wedge B \vdash_{fd} \Delta \quad \Gamma \vdash_{fd} \Delta, A \quad \Gamma \vdash_{fd} \Delta, B}{\Gamma \vdash_{fd} \Delta} [\wedge GI]$$

$$\frac{\Gamma, \neg A \vdash_{fd} \Delta \quad \Gamma, A \vdash_{fd} \Delta}{\Gamma \vdash_{fd} \Delta} [\neg GI_1] \qquad \frac{\Gamma \vdash_{fd} \Delta, \neg A \quad \Gamma \vdash_{fd} \Delta, A}{\Gamma \vdash_{fd} \Delta} [\neg GI_2]$$

Assign all meta-level atoms with negative polarity:

$$\lfloor B \rfloor^{\perp} \to \lceil B \rceil \qquad \lceil B \rceil^{\perp} \to \lfloor B \rfloor$$

An adequacy on the level of derivations can also be obtained.

Parigot's notion of “killing" a premise is handled by polarities.

Other proof systems

In the paper, we also deal with:

- the KE tableaux of D'Agostino and Mondadori, and
- a proof system of Smullyan with many axioms and with cut as the only inference rule.

Agenda

- Overview
- Linear Logic and Focusing
- Encoding Systems
- Results
- **Conclusions and Future Works**

Conclusions and Future Works

We have worked with essentially one ``definition'' of the two senses of a logical connective.

We allowed either changes in polarity assignment to atoms or replacing specifications with logically equivalent formulas.

This simple meta-level tuning accounts faithfully for a number of (object-level) proof systems.

Classical systems can usually be encoded with an adequacy to the level of derivations, while intuitionistic systems are encoded only to the level of proofs.

There is a conflict between uses of exponentials to improve adequacy of encodings and the focusing discipline that is at the heart of getting adequacy results in the first place. More work on modals? focusing?