Relating Focused Proofs with Different Polarity Assignments

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Abstract

In this work, we will reason on how a given focused proof where atoms are assigned with some polarity can be transformed into another focused proof where the polarity assignment to atoms is changed. This will allow, in principle, transforming a proof obtained using one proof system into a proof using another proof system. More specifically, using the intuitionistic focused system LJF restricted to Harrop formulas, we define a procedure, introducing cuts, for transforming a focused proof where an atom is assigned with positive polarity into another focused proof where the same atom is assigned negative polarity and vice-versa. Then we show how to eliminate these cuts, obtaining a very interesting result: while the process of eliminating a cut on a positive atom gives rise to a proof with one smaller cut, in the negative case the number of introduced cuts grows exponentially. This difference in the cut-elimination algorithm seems to be related to the different evaluation strategies according to the Curry-Howard isomorphism, where cut-elimination corresponds to computation in a functional programming setting.

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1. Introduction

In focused proof systems, such as Andreoli's original focused proof system [1] for linear logic or Liang and Miller's LJF and LKF focused proof systems for intuitionistic and classical logics [5], connectives are classified as positive or negative, according to their right introduction rules: positive connectives have not necessarily invertible rules, while negative connectives are those whose right introduction rules are invertible. The polarity of a non atomic formula is then given by the polarity of its outermost connective. The interesting fact is that atomic formulas can be arbitrarily assigned as positive or negative, without affecting the completeness of the focusing discipline.

While this choice for the polarity of atomic formulas does not affect provability, it does affect the shape of the resulting focused proofs obtained. For instance, it has been shown that this choice can explain different proof search strategies, such as backward chaining and forward chaining [2, 5]. For another example, Nigam and Miller have shown in [8] that depending on the polarity assignments used for the atomic formula, one can, from the same logical theory, encode sequent calculus proofs or natural deduction ones.

In this paper, we show *how* a given focused proof where atoms are assigned with some polarity assignment can be transformed into another focused proof where the polarity assignment to atoms is changed. Hence, one could be able to transform a proof using a forward chaining strategy into a proof using backward chaining strategy or even obtain (possibly novel) translations from Sequent Calculus to Natural Deduction and vice versa.

More specifically, using the focused system LJF [5] for intuitionistic logic restricted to hereditarry Harrop formulas [6], we define a procedure, introducing cuts, for transforming a focused proof where an atom is assigned with positive polarity into another focused proof where the same atom is assigned negative polarity and vice-versa. Then we show how to eliminate these cuts. Interestingly, while the process of eliminating a cut on a positive atom gives rise to a proof with one smaller cut, in the negative case the number of introduced cuts grows exponentially.

This difference in the cut-elimination algorithm seems to be related to the different evaluation strategies according to the Curry-Howard isomorphism, where cut-elimination corresponds to computation in a functional programming setting. In particular, we will show that how the polarities of atoms is assigned is related to Call-by-Value and Call-by-Name reduction strategies. This means that in principle one could obtain in the same proof system a mixture of both call-by-value and call-by-name reduction strategies by simply playing with the polarity assignment of atomic formulas.

This is an ongoing work paper. Hence some ideas are not yet fully developed, specially those on Section 8. The paper is organized as follows. In Section 2 we recall the Focused Proof System for Intuitionistic Logic LJF, proposed in [5]. Section 3 presents the fragment we will be working on and in Section 4 we show how to transform proofs from one polarity to another. In Section 5 we show how to eliminate the cuts introduced during the polarities' transformation process. We define the typed λ-calculus system λ_{LJF} in Section 6 and show all its possible cut reductions in Section 7. Finally, in Section 8 we show how the cut elimination procedure is related to the different evaluation strategies while Section 9 concludes the paper.

2. The Focused Proof System LJF for Intuitionistic Logic

In order to present the focused proof system LJF, we first classify the connectives $\wedge^+$, $\exists$, $true$ and $\bot$ as *positive* (their right introduction is not necessarily invertible) and the connectives $\supset$, and $\forall$ as *negative* (their right introduction rules are invertible). This dichotomy must also be extended to formulas. Concerning the atomic ones: some pre-chosen atoms are considered negative and the rest are considered positive. That is, one is free to assign as positive or negative the polarity to atoms. From this, a formula is positive if its main connective is positive or it is a *positive atom* and is negative if its main connective is negative or it is a *negative atom*.

The proof system LJF depicted in Figure 1 has four types of sequents.

1. The sequent $[\Gamma]-_A\!\rightarrow$ is a *right-focusing* sequent (the focus is A);
2. The sequent $[\Gamma] \xrightarrow{A} [R]$: is a *left-focusing* sequent (with focus on A);
3. The sequent $[\Gamma], \Theta \longrightarrow \mathcal{R}$ is an *unfocused sequent*. Here, Γ contains negative formulas and positive atoms, and $\mathcal{R}$ is either in brackets, written as $[R]$, or without brackets;
4. The sequent $[\Gamma] \longrightarrow [R]$ is an instance of the previous sequent where Θ is empty.

As an inspection of the inference rules of LJF reveals, the search for a *focused* proof is composed of two alternating phases, and these phases are governed by polarities. The *negative phase* applies invertible (negative) rules until exhaustion: no backtracking during this phase of search is needed. The negative phase uses the third type of sequent above (the unfocused sequents): in that case, Θ contains positive or negative formulas. If Θ contains positive formulas, then an introduction rule (either $\wedge_l, \exists_l$, $true_l$, or $false_l$) is used to decompose it; if it is negative, then the formula is moved to the Γ context (by using the $[]_l$ rule). The end of the negative phase is represented by the fourth type of sequent. Such a sequent is then established by using one of the decide rules, D_r or D_l. The application of one of these decide rules then selects a formula for focusing and switches proof search to the *positive phase* or *focused phase*. This focused phase then proceeds by applying sequences of inference rules on focused formulas: in general, backtracking may be necessary in this phase of search. The focusing phase ends with one of the *release rule* R_l or R_r.

As is pointed out in [4], if all atoms are given negative polarity, the resulting proof system models backward chaining proof search and uniform proofs [6]. If positive atoms are permitted as well, then forward chaining steps can also be accommodated. Moreover, as in [8], it is possible in LJF to specify with the same intuitionistic theory sequent calculus proofs by using one polarity assignment and natural deduction proofs by using another polarity assignment.

3. Logic programming fragment

For this paper, we restrict theories used to be the D-formulas and goals to be the G-formulas both specified by the grammar below:

$$
\begin{aligned}
N &:= A \mid N_1 \wedge^- N_2 \mid G \supset N \mid \forall x\, N\\
G &:= true \mid A \mid G_1 \wedge^+ G_2 \mid G_1 \vee G_2 \mid \exists x\, G \mid \forall x\, G \mid D \supset G\\
D &:= A \mid N \mid D_1 \wedge^+ D_2 \mid \exists x\, D
\end{aligned}
$$

This is a straightforward extension of the fragment of hereditary Harrop formulas used to describe uniform proofs [6].

We restrict our language to this fragment mainly for presentation reasons, as it considerably simplifies the machinery used in the following sections. In particular, it allows for a concise cut-elimination procedure involving only some cut permutations shown in Section 5, which will be used in the subsequent Sections to demonstrate the connections of the polarity assignment to Call-by-Value and Call-by-Name reduction strategies. It seems possible to repeat the results in Sections 4 for the whole logic, but that would require in further reduction cases.

An important property of cut-free LJF proofs of any sequent containing only D formulas on the left and a G formula on the right is that there are no occurrences of $\vee_l$ nor of $\wedge_r^-$ rules. Moreover, the following is also true:

Lemma 3.1. *Let Γ be a set of D-formulas. Let Ξ be a positive trunk, that is a derivation containing only rules from the positive phase, with end sequent of the form $[\Gamma]-_F\!\rightarrow$, then there is no sequent focused on the left in Ξ.*

Proof The proof follows by the induction on the height of the positive trunk. In particular, only the positive rules $I_r, \wedge_r^+, \vee_r, true_r$ and $\exists_r$ are applicable and all these rules do not lose right focus. $\square$

4. Changing polarities

In this section, we show how to transform focused proof where an atom is assigned with one polarity to a focused proof where this same atom is assigned the opposite polarity. The transformations below might not preserve the size of a proof. In fact, it may well happen that after a proof is transformed from one proof system to another, the proof increases exponentially. Although this is relevant in some cases, such as in Proof Carrying Code, it is not that relevant when trying to unify the library of results obtained with different proof systems.

4.1 From positive polarity to negative polarity

In this section we demonstrate how to transform a focused proof where an atom is assigned with positive polarity into another focused proof where the same atom is assigned negative polarity. Assume that Ξ is a proof where the atom A is assigned with positive polarity. We modify Ξ by induction from the leaves to the root on the number of reaction left and initial right rules applied on A. In particular, we perform the following operations:

The base case is when the proof ends with an initial right rule, which can only appear in positive derivations. We eliminate initial right rules by replacing the following subderivations appearing in a positive derivation:

$$
\dfrac{}{[\Gamma]-_A\!\rightarrow}\;I_r \quad\text{and}\quad \dfrac{\dfrac{}{[\Gamma]-_A\!\rightarrow}\;I_r}{[\Gamma]\longrightarrow[A]}\;D_r
$$

by the following derivations, respectively:

$$
\dfrac{\dfrac{\dfrac{}{[\Gamma]\xrightarrow{A}[A]}\;I_l}{[\Gamma]\longrightarrow[A]}\;D_l}{[\Gamma]-_A\!\rightarrow}\;R_r,[]_r \quad\text{and}\quad \dfrac{\dfrac{}{[\Gamma]\xrightarrow{A}[A]}\;I_l}{[\Gamma]\longrightarrow[A]}\;D_l
$$

Notice that from the former derivations, it is the case that $A \in \Gamma$ and therefore we can, in the latter derivations, focus on A.

The other possible cases are when one of the rules $\supset_l$, $\wedge_l^-$ or $\forall_l$ are applied. In those cases, an instance of the cut rule is added. We illustrate the case of $\supset_l$, the others are similar and simpler.

$$
\dfrac{\dfrac{\begin{matrix}\Xi_1\\ [\Gamma]-_G\!\rightarrow\end{matrix} \quad \dfrac{\begin{matrix}\Xi_2\\ [\Gamma,A]\longrightarrow[G']\end{matrix}}{[\Gamma]\xrightarrow{A}[G']}\;R_l,[]_l}{[\Gamma]\xrightarrow{G\supset A}[G']}\;\supset_l}{[\Gamma]\longrightarrow[G']}\;D_l
\quad\Longrightarrow\quad
\dfrac{\dfrac{\dfrac{\begin{matrix}\Xi_1'\\ [\Gamma]-_G\!\rightarrow\end{matrix} \quad \dfrac{}{[\Gamma]\xrightarrow{A}[A]}\;I_l}{[\Gamma]\xrightarrow{G\supset A}[A]}\;\supset_l}{[\Gamma]\longrightarrow A}\;[]_r D_l \quad \begin{matrix}\Xi_2'\\ [\Gamma,A]\longrightarrow[G']\end{matrix}}{[\Gamma]\longrightarrow[G']}\;cut
$$

Here, the derivations Ξ_1' and Ξ_2' are obtained by applying the inductive hypothesis to Ξ_1 and Ξ_2 of smaller height and transforming all occurrences of A with positive polarity into negative polarity. Notice that, from Lemma 3.1, in the remaining of positive trunk in Ξ_1 there may not be any occurrences of reaction left rules, but only of initial right rules which are handled by the base case. Hence, this operation removes all reaction left rules over all the appearances of the atomic formula A.

Finally, after applying these operations, we obtain an LJF proof with cuts. To obtain a cut-free proof, we apply the cut-elimination

$$\frac{[N,\Gamma] \xrightarrow{N} [R]}{[N,\Gamma] \longrightarrow [R]}\ D_l \qquad \frac{[\Gamma]{-}_{P}{\rightarrow}}{[\Gamma] \longrightarrow [P]}\ D_r \qquad \frac{[\Gamma], P \longrightarrow [R]}{[\Gamma] \xrightarrow{P} [R]}\ R_l \qquad \frac{[\Gamma] \longrightarrow N}{[\Gamma]{-}_{N}{\rightarrow}}\ R_r$$

$$\frac{[\Gamma, N_a], \Theta \longrightarrow \mathcal{R}}{[\Gamma], \Theta, N_a \longrightarrow \mathcal{R}}\ []_l \qquad \frac{[\Gamma], \Theta \longrightarrow [P_a]}{[\Gamma], \Theta \longrightarrow P_a}\ []_r \qquad \frac{}{[\Gamma] \xrightarrow{A_n} [A_n]}\ I_l \qquad \frac{}{[\Gamma, A_p]{-}_{A_p}{\rightarrow}}\ I_r$$

$$\frac{}{[\Gamma], \Theta, \bot \longrightarrow \mathcal{R}}\ \mathit{false}_l \qquad \frac{[\Gamma], \Theta \longrightarrow \mathcal{R}}{[\Gamma], \Theta, \mathit{true} \longrightarrow \mathcal{R}}\ \mathit{true}_l \qquad \frac{}{[\Gamma]{-}_{\mathit{true}}{\rightarrow}}\ \mathit{true}_r$$

$$\frac{[\Gamma], \Theta, A, B \longrightarrow \mathcal{R}}{[\Gamma], \Theta, A \wedge^+ B \longrightarrow \mathcal{R}}\ \wedge_l^+ \qquad \frac{[\Gamma]{-}_{A}{\rightarrow} \quad [\Gamma]{-}_{B}{\rightarrow}}{[\Gamma]{-}_{A\wedge^+ B}{\rightarrow}}\ \wedge_r^+ \qquad \frac{[\Gamma]{-}_{A}{\rightarrow} \quad [\Gamma] \xrightarrow{B} [R]}{[\Gamma] \xrightarrow{A \supset B} [R]}\ \supset_l \qquad \frac{[\Gamma], \Theta, A \longrightarrow B}{[\Gamma], \Theta \longrightarrow A \supset B}\ \supset_r$$

$$\frac{[\Gamma] \xrightarrow{A_i} [R]}{[\Gamma] \xrightarrow{A_1 \wedge^- A_2} [R]}\ \wedge_{li}^- \qquad \frac{[\Gamma], \Theta \longrightarrow A \quad [\Gamma], \Theta \longrightarrow B}{[\Gamma], \Theta \longrightarrow A \wedge^- B}\ \wedge_r^- \qquad \frac{[\Gamma], \Theta, A \longrightarrow \mathcal{R} \quad [\Gamma], \Theta, B \longrightarrow \mathcal{R}}{[\Gamma], \Theta, A \vee B \longrightarrow \mathcal{R}}\ \vee_l \qquad \frac{[\Gamma]{-}_{A_i}{\rightarrow}}{[\Gamma]{-}_{A_1 \vee A_2}{\rightarrow}}\ \vee_{ri}$$

$$\frac{[\Gamma], \Theta, A \longrightarrow \mathcal{R}}{[\Gamma], \Theta, \exists y A \longrightarrow \mathcal{R}}\ \exists_l \qquad \frac{[\Gamma]{-}_{A[t/x]}{\rightarrow}}{[\Gamma]{-}_{\exists x A}{\rightarrow}}\ \exists_r \qquad \frac{[\Gamma] \xrightarrow{A[t/x]} [R]}{[\Gamma] \xrightarrow{\forall x A} [R]}\ \forall_l \qquad \frac{[\Gamma], \Theta \longrightarrow A}{[\Gamma], \Theta \longrightarrow \forall y A}\ \forall_r$$

Figure 1. The LJF system [5]. Here A_n denotes a negative atom, A_p a positive atom, P a positive formula, N a negative formula, N_a a negative formula or an atom, and P_a a positive formula or an atom. All other formulas are arbitrary and y is not free in Γ, Θ or R.

theorem given in Section 5. The resulting proof is a cut-free focused proof where the polarity of the atom A is negative.

4.2 From negative to positive polarity

The idea to transform a proof where an atom A is assigned with negative polarity to a proof where the same atom appears with positive polarity is similar to the previous case. We perform the following operations to the original proof:

$$\frac{\dfrac{\dfrac{}{[\Gamma] \xrightarrow{A} [A]}\ I_l \qquad \overset{\Xi}{[\Gamma]{-}_{G}{\rightarrow}}}{[\Gamma] \xrightarrow{G \supset A} [A]}\ \supset_l}{[\Gamma] \longrightarrow [A]}\ D_l \quad \Longrightarrow \quad \frac{\dfrac{\dfrac{\dfrac{\dfrac{}{[\Gamma, A]{-}_{A}{\rightarrow}}\ I_r}{[\Gamma, A] \longrightarrow [A]}\ D_r}{[\Gamma] \xrightarrow{A} [A]}\ R_l, []_l \qquad \overset{\Xi'}{[\Gamma]{-}_{G}{\rightarrow}}}{[\Gamma] \xrightarrow{G \supset A} [A]}\ \supset_l}{[\Gamma] \longrightarrow [A]}\ D_l$$

To eliminate all occurrences of R_r, we will make use of the cut rule. Consider the following positive derivation containing R_r rules on the negative polarity atom A and whose last rule is D_r:

$$\frac{\dfrac{\overset{\Xi_1}{[\Gamma]{-}_{G_1}{\rightarrow}} \quad \cdots \quad \dfrac{\overset{\Xi_i}{[\Gamma] \longrightarrow A}}{[\Gamma]{-}_{A}{\rightarrow}}\ R_r \quad \cdots \quad \overset{\Xi_n}{[\Gamma]{-}_{G_n}{\rightarrow}}}{[\Gamma]{-}_{G}{\rightarrow}}}{[\Gamma] \longrightarrow [G]}$$

It can be transformed to the following derivation where A, where the number of reaction rules is reduced and this occurrence of A has positive polarity.

$$\frac{\overset{\Xi_i'}{[\Gamma] \longrightarrow A} \qquad \dfrac{\dfrac{\overset{\Xi_1'}{[\Gamma, A]{-}_{G_1}{\rightarrow}} \ \cdots \ \dfrac{}{[\Gamma, A]{-}_{A}{\rightarrow}}\ I_r \ \cdots \ \overset{\Xi_n'}{[\Gamma, A]{-}_{G_n}{\rightarrow}}}{[\Gamma, A]{-}_{G}{\rightarrow}}}{[\Gamma, A] \longrightarrow [G]}}{[\Gamma] \longrightarrow [G]}\ \mathit{cut}$$

The proofs $\Xi_1', \ldots, \Xi_n'$ are obtained by applying the inductive hypothesis where A has positive polarity. The inductive hypothesis is applicable since their height are smaller and the number of reaction rules is decreased by at least one.

5. Cut-elimination

5.1 If cut-formula is a positive atom

Instead of using the cut-elimination algorithm with several intra-phase cut-rules given in [5], we exploit the fact that the theories encoding proof systems are hereditary Harrop formulas to give a simpler cut-elimination procedure, with only inter-phase cut-rules. In particular, our algorithm consists of basically two rewrite rules, depending on which decide rule is applied last on left premise of the cut rule. If it is D_r then it is necessarily the case that the atom A used in the cut is in the context Γ, which implies that the cut is not necessary:

$$\frac{\dfrac{\dfrac{}{[\Gamma]{-}_{A}{\rightarrow}}}{[\Gamma] \longrightarrow A}\ []_r, D_r \qquad \overset{\Xi}{[\Gamma, A] \longrightarrow [G]}}{[\Gamma] \longrightarrow [G]}\ \mathit{cut}$$

This derivation reduces to the following derivation where the cut is eliminated:

$$[\Gamma] \overset{\Xi}{\longrightarrow} [G]$$

For the second case, when the decide rule D_l is applied last in the left premise of the cut rule, we proceed as follows:

$$\frac{\dfrac{\dfrac{[\Gamma_1]{-}_{B_1}{\rightarrow} \quad \cdots \quad [\Gamma_n]{-}_{B_n}{\rightarrow} \qquad \dfrac{\dfrac{\overset{\Xi_1}{[\Gamma, A'] \longrightarrow [A]}}{[\Gamma], A' \longrightarrow [A]}}{[\Gamma] \xrightarrow{A'} [A]}\ R_l}{[\Gamma] \xrightarrow{F} [A]}}{[\Gamma] \longrightarrow A}\ []_r, D_l \qquad \overset{\Xi_2}{[\Gamma, A] \longrightarrow [G]}}{[\Gamma] \longrightarrow [G]}\ \mathit{cut}$$

Since our theories are hereditary Harrop formulas, once the formula F is focused on, the resulting formula focused on the left is necessarily an atom. Moreover, the atom A' cannot be negative otherwise one would have to finish the proof with an I_l rule, but this is not possible since the atom appearing at the right-hand-side, A, is positive. Hence, it is necessarily the case that the atom A' is positive and since it is focused on the left, one releases focus.

We permute the atomic cut above the positive phase to the left as follows:

$$\dfrac{[\Gamma_1]-_{B_1}\rightarrow \quad \cdots \quad [\Gamma_n]-_{B_n}\rightarrow \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overset{\Xi_1}{[\Gamma, A'] \longrightarrow [A]}}{[\Gamma, A'] \longrightarrow A}\ []_r \quad \overset{\Xi_2}{[\Gamma, A, A'] \longrightarrow [G]}}{[\Gamma, A'] \longrightarrow [G]}\ Cut}{[\Gamma], A' \longrightarrow [G]}}{[\Gamma] \xrightarrow{A'} [G]}\ R_l}{\dfrac{[\Gamma] \xrightarrow{F} [G]}{[\Gamma] \longrightarrow G}\ []_r, D_l}$$

5.2 If cut-formula is a negative atom

It turns out that the cut may not permute upwards on the left premise if A is negative. In fact, on focusing on a left formula F like in the last Section, if the resulting atom focusing on the left is negative, it has necessarily to be A and the proof finishes with an I_l rule. For all other cases we could proceed like in the positive case.

There are two base cases:

$$\dfrac{\overset{\Xi}{[\Gamma] \longrightarrow [A]} \quad \dfrac{\dfrac{}{[\Gamma, A] \xrightarrow{A} [A]}\ I_l}{[\Gamma, A] \longrightarrow [A]}\ D_l}{[\Gamma] \longrightarrow [A]}\ cut \qquad \Longrightarrow \qquad [\Gamma] \overset{\Xi}{\longrightarrow} [A]$$

$$\dfrac{\overset{\Xi}{[\Gamma] \longrightarrow [A]} \quad \dfrac{\dfrac{}{[\Gamma, A] \xrightarrow{A'} [A']}\ I_l}{[\Gamma, A] \longrightarrow [A']}\ D_l}{[\Gamma] \longrightarrow [A']}\ cut \qquad \Longrightarrow \qquad \dfrac{\dfrac{}{[\Gamma] \xrightarrow{A'} [A']}\ I_l}{[\Gamma] \longrightarrow [A']}\ D_l$$

The inductive cases are obtained by moving the cut rule upwards.

Let $\star$ be the maximum sequence of inference rules excluding decide rules appearing above the sequent $[\Gamma, A] \longrightarrow [G]$ (hence $\star$ has only negative rules). Let n be the minimum length of the sub-derivations of $\star$. If $n > 0$,

$$\dfrac{\overset{\Xi}{[\Gamma] \longrightarrow [A]} \quad \dfrac{\overset{\Xi'}{[\Gamma', A] \longrightarrow [G']}}{[\Gamma, A] \longrightarrow [G]}\ \star}{[\Gamma] \longrightarrow [G]}\ cut$$

where $\Gamma \subseteq \Gamma'$.

If, on the other hand, $n = 0$, the last rule applied for proving $[\Gamma, A] \longrightarrow [G]$ is a decision rule. There are then two sub-cases: D_l and D_r.

In both cases, after finishing the focus phases (positive or negative) we will end up with a proof of the shape (ignoring the leaves):

$$\dfrac{\overset{\Xi}{[\Gamma] \longrightarrow [A]} \quad \dfrac{\overset{\Xi_1}{[\Gamma_1, A] \longrightarrow [G_1]} \quad \cdots \quad \overset{\Xi_n}{[\Gamma_n, A] \longrightarrow [G_n]}}{[\Gamma, A] \longrightarrow [G]}}{[\Gamma] \longrightarrow [G]}\ cut$$

and the cut is moved upwards as follows:

$$\dfrac{\dfrac{\overset{\Xi}{[\Gamma] \longrightarrow [A]} \quad \overset{\Xi_1}{[\Gamma_1, A] \longrightarrow [G_1]}}{[\Gamma_1] \longrightarrow [G_1]}\ cut \quad \cdots \quad \dfrac{\overset{\Xi}{[\Gamma_n] \longrightarrow [A]} \quad \overset{\Xi_1}{[\Gamma_n, A] \longrightarrow [G_n]}}{[\Gamma_n] \longrightarrow [G_n]}\ cut}{[\Gamma] \longrightarrow [G]}$$

Observe that, in this case, the size of proof grows exponentially.

6. The λ_{LJF} calculus

It is interesting to notice in the previous Sections that the cut-elimination algorithm is forced to move in one direction or to the other according to the polarity of atoms. This seems to be related to the different evaluation strategies according to the Curry-Howard isomorphism, where cut-elimination corresponds to computation in a functional programming setting. In particular, we will try to show that how the polarities of atoms is assigned is related to call-by-value and call-by-name reduction strategies.

Assume that we are in an implicational fragment of logic. That is formulas are F-formulas in the following grammar:

$$F ::= A \mid F_1 \supset F_2$$

where A is an atomic formula.

We will assume the following sorts of terms:

$$\begin{array}{l}
V^+ ::= \{x^+\} \qquad V^- ::= \{x^-\}^{x^-} \\
P ::= y(R_1, \ldots, R_n, L^x) \mid C \\
N ::= \lambda x_1 \cdots x_n.D \\
D ::= V^+ \mid V^- \mid P \\
R ::= V^+ \mid \{N\} \mid \{P\} \\
L ::= V^- \mid V^+ \mid \{P\}^{x^+} \\
C ::= cut(N, x.D) \mid cut(D, x.D)
\end{array}$$

Observe that there are two sorts of atomic variables: V^- and V^+. We make sure that the type of the variable matches the assignment of polarity of the corresponding associated atomic type. That is, if $x \in V^-$ then its associated atomic type is negative. Similarly, if $x \in V^+$ then its associated atomic type is positive.

We can now annotate terms with formulas and define an inference system for them. We will call the resulting system the λ_{LJF} calculus. Following the focusing behavior, the logical rules can be reduced to the two "macro-rules" introducing a formula F.

$$\dfrac{\dfrac{\Gamma, y : F -_{R_1:F_1}\rightarrow \qquad \cdots \qquad \Gamma, y : F -_{R_n:F_n}\rightarrow \qquad \Gamma, y : F \xrightarrow{x:A_{n+1}} L : A}{\Gamma, y : F \xrightarrow{y:F} y(R_1, \ldots, R_n, L^x) : A}}{\Gamma, y : F \longrightarrow y(R_1, \ldots, R_n, L^x) : A}$$

$$\dfrac{\Gamma, x_1 : F_1, \ldots, x_n : F_n \longrightarrow D : A_{n+1}}{\Gamma \longrightarrow \lambda x_1 \cdots x_n.D : F}$$

Here, A_{n+1} is the atomic formula appearing at the right-most branch of the syntax tree of F. The remaining relevant rules are also annotated with terms as follows:

$$\dfrac{}{\Gamma \xrightarrow{x^-:A_n} \{x^-\}^{x^-} : A_n}\ I_l \qquad \dfrac{}{\Gamma, x^+ : A_p -_{\{x^+\}:A_p}\rightarrow}\ I_r^1$$

$$\dfrac{}{\Gamma, x^+ : A_p \longrightarrow \{x^+\} : A_p}\ I_r^2 \qquad \dfrac{\Gamma, x^+ : A_p \longrightarrow P : A'}{\Gamma \xrightarrow{x^+:A_p} \{P\}^{x^+} : A'}\ R_l$$

$$\dfrac{\Gamma \longrightarrow P : A_n}{\Gamma -_{\{P\}:A_n}\rightarrow}\ R_r^1 \qquad \dfrac{\Gamma \longrightarrow N : F_1 \supset F_2}{\Gamma -_{\{N\}:F_1 \supset F_2}\rightarrow}\ R_r^2$$

$$\dfrac{\Gamma \longrightarrow N : F_1 \supset F_2 \quad \Gamma, x : F_1 \supset F_2 \longrightarrow D : A}{\Gamma \longrightarrow cut(N, x.D) : A}\ Cut^1$$

$$\dfrac{\Gamma \longrightarrow D : A_1 \quad \Gamma, x : A_1 \longrightarrow D' : A}{\Gamma \longrightarrow cut(D, x.D') : A}\ Cut^2$$

Some observations: If all atoms are negative, then there are no occurrences of R_l nor of I_r. Hence all y-terms are of the form: $y(R_1, \ldots, R_n, V^-)$. On the other hand, if all atoms are positive, then all y-terms are of the form: $y(R_1, \ldots, R_n, \{P\}^{x^+})$.

7. Cut reductions

There are a number of cut-reductions to consider. As before, we use inter-phase cuts.

Case Cut^1

There are basically three cases to consider.

1. If the cut term is principal in both premises and not atomic

$$\dfrac{\dfrac{\Gamma, x_1 : A_1, \ldots, x_n : A_n \longrightarrow D : A_{n+1}}{\Gamma \longrightarrow \lambda x_1 \cdots x_n.D : F} \quad \dfrac{\Gamma, y : F -_{R_1:A_1}\rightarrow \quad \cdots \quad \Gamma, y : F \overset{x_{n+1}:A_{n+1}}{\longrightarrow} L : A}{\Gamma, y : F \longrightarrow y(R_1, \ldots, R_n, L^{x_{n+1}}) : A}}{\Gamma \longrightarrow cut(\lambda x_1 \cdots x_n.D, y.y(R_1, \ldots, R_n, L^{x_{n+1}}) : A} \; Cut^1$$

If $y \notin FV(R_1) \cup \cdots \cup FV(R_n) \cup FV(L)$, then the following reduction is valid.

$$\dfrac{\Gamma \longrightarrow R'_1 : A_1 \quad \cdots \quad \Gamma \longrightarrow R'_n : A_n \quad \dfrac{\Gamma' \longrightarrow D : A_{n+1} \quad \Gamma', x_{n+1} : A_{n+1} \longrightarrow L' : A}{\Gamma, x_1 : A_1, \ldots, x_n : A_n \longrightarrow cut(D, x_{n+1}.L' : A)}}{\Gamma \longrightarrow cut(R'_1, x_1.cut(\cdots cut(R'_n, x_n.cut(D, x_{n+1}.L') \cdots) : A}$$

where $\Gamma' = \Gamma, x_1 : A_1, \ldots, x_n : A_n$; $R_i = R'_i$ if $R_i \in V^+$ and $R_i = \{R'_i\}$ if $R_i \notin V^+$; and $L' = L$ if $L \in V^-$ and $L = \{L\}^{x^+_{n+1}}$ if $L \notin V^-$.

This reduction corresponds to the rewrite:

Red1 : $cut(\lambda x_1 \cdots x_n.D, y.y(R_1, \ldots, R_n, L^{x_{n+1}}) \longrightarrow$
$cut(R'_1, x_1.cut(\cdots cut(R'_n, x_n.cut(D, x_{n+1}.L') \cdots)$

2. If the cut term in the right premise is principal and atomic. In this case the cut can be eliminated and the rewrite rule is the trivial one:

$$\dfrac{\overset{\Pi_1}{\Gamma \longrightarrow \lambda x_1 \cdots x_n.D : F} \quad \overset{\Pi_2}{\Gamma, y : F \longrightarrow V : A}}{\Gamma \longrightarrow cut(\lambda x_1 \cdots x_n.D, y.V) : A} \; Cut^1$$

If V is principal, then $V \in \Gamma$, Π_2 is the initial axiom (maybe preceded by R_l) and the whole proof can be substituted by Π_1.

Then we have the (trivial) rewrite:

Red1′ : $cut(\lambda x_1 \cdots x_n.D, y.V) \longrightarrow \lambda x_1 \cdots x_n.D$

3. The cases involving inner cuts, that is, when $D = C$, are handled as usual, i.e., reducing first the deeper cut.

Case Cut^2

1. If the cut term in the left premise is principal and not cut or atomic:

$$\dfrac{\dfrac{\Gamma, y : F -_{R_1:A_1}\rightarrow \quad \cdots \quad \Gamma, y : F \overset{x_{n+1}:A_{n+1}}{\longrightarrow} L : A}{\Gamma, y : F \longrightarrow y(R_1, \ldots, R_n, L^{x_{n+1}}) : A} \quad \Gamma, y : F, x : A \longrightarrow D : A'}{\Gamma, y : F \longrightarrow cut(y(R_1, \ldots, R_n, L^{x_{n+1}}), x.D) : A'} \; Cut^2$$

If $x_{n+1} \in V^+$ the following reduction is valid (where $\Gamma' = \Gamma \cup y : F$):

$$\dfrac{\Gamma, y : F -_{R_1:A_1}\rightarrow \quad \cdots \quad \Gamma' -_{R_n:A_n}\rightarrow \quad \dfrac{\dfrac{\Gamma', x_{n+1} : A_{n+1} \longrightarrow L : A \quad \Gamma', x_{n+1} : A_{n+1}, x : A \longrightarrow D : A'}{\Gamma', x_{n+1} : A_{n+1} \longrightarrow cut(L, x.D) : A'}}{\Gamma' \overset{x_{n+1}:A_{n+1}}{\longrightarrow} \{cut(L, x.D)\}^{x_{n+1}} : A'}}{\Gamma' \longrightarrow y(R_1, \ldots, R_n, \{cut(L, x.D)\}^{x_{n+1}}) : A'}$$

This reduction corresponds to the rewrite:

Red2 : $cut(y(R_1, \ldots, R_n, L^{x_{n+1}}), x.D) \longrightarrow$
$y(R_1, \ldots, R_n, \{cut(L, x.D)\}^{x_{n+1}})$

If $x_{n+1} \in V^-$, then $L = \{x^-_{n+1}\}^-$ and $A_{n+1} = A$. In this case, the cut can be eliminated and the reduction corresponds to the rewrite:

Red2′ : $cut(y(R_1, \ldots, R_n, L^{x_{n+1}}), x.D) \longrightarrow$
$y(R_1, \ldots, R_n, x.D)$

2. If the cut term in the left premise is principal and atomic, as in the case for Cut^1, it can be eliminated and the rewrite rule is the trivial one:

$$\dfrac{\overset{\Pi_1}{\Gamma \longrightarrow V : A_1} \quad \overset{\Pi_2}{\Gamma, x : A_1 \longrightarrow D}}{\Gamma \longrightarrow cut(V, x.D) : A} \; Cut^2$$

If V is principal, then $V \in \Gamma$, Π_1 is the initial axiom (maybe preceded by R_l) and the whole proof can be substituted by Π_2.

Then we have the (trivial) rewrite:

Red2″ : $cut(V, x.D) \longrightarrow D$

3. If the cut-term in the left premise is a cut, then $D = C$ and we reduce first the deeper cut.

8. Cut reductions and λ-calculus

We will give here an idea on how we intend to relate polarities and cut-reduction. This is really ongoing work, so we will not provide proofs, only ideas.

We will start by the call-by-value λ-calculus. We will use here Moggi's λ_C [7], where terms are defined as

$$M, N, P \quad ::= \quad x \mid \lambda x.M \mid MN \mid \texttt{let}\ x = M\ \texttt{in}\ N$$

The type system is the standard one, with only atomic and functional types.

We use V as a meta-variable ranging only over values. The reduction rules of λ_C are as follows:

$(\lambda x.M)V$	$\longrightarrow$	$Mx = V$
$\texttt{let}\ x = V\ \texttt{in}\ M$	$\longrightarrow$	$Mx = V$
MN	$\longrightarrow$	$\texttt{let}\ x = M\ \texttt{in}\ (xN)$
VN	$\longrightarrow$	$\texttt{let}\ y = N\ \texttt{in}\ (Vy)$
$\texttt{let}\ x = M\ \texttt{in}\ x$	$\longrightarrow$	M
$\texttt{let}\ y = (\texttt{let}\ x = M\ \texttt{in}\ N)\ \texttt{in}\ P$	$\longrightarrow$	$\texttt{let}\ x = M\ \texttt{in}\ (\texttt{let}\ y = N\ \texttt{in}\ P)$

Consider now the λ_{LJF} calculus, described in Section 6, restricted to *positive* variables. Hence we have to restrict the cut, so that we will only use the Cut^2 rule.

We define a translation t from λ_{LJF} to λ_C as:

x^t	$=$	x
$(\{M\})^t$	$=$	$(M)^t$
$(\lambda x_1 \ldots x_n.M)^t$	$=$	$\lambda x_1 \ldots x_n.(M)^t$
$(y(R_1, \ldots, R_n, L^x))^t$	$=$	$\texttt{let}\ x = y(R_1)^t \ldots (R_n)^t\ \texttt{in}\ (L)^t$
$(cut(D, x.D'))^t$	$=$	$\texttt{let}\ x = (D)^t\ \texttt{in}\ (D')^t$

We claim that:

1. For any proof terms M and N, $M \longleftrightarrow^* N$ iff $M^t \longleftrightarrow^* N^t$.
2. The cut reductions derived in Section 7 for λ_{LJF} strongly correspond to the call-by-name reduction in λ_C.

On the other hand, if we restrict our system to *negative* variables, we have the following translation of our system in the call-by-name λ-calculus:

x^t	$=$	x
$(\{M\})^t$	$=$	$(M)^t$
$(\lambda x_1 \ldots x_n.M)^t$	$=$	$\lambda x_1 \ldots x_n.(M)^t$
$(y(R_1, \ldots, R_n, L^x))^t$	$=$	$(L)^t\{x = y(R_1)^t \ldots (R_n)^t\}$
$(cut(N, x.D))^t$	$=$	$(D)^t\{x = (N)^t\}$

The claims, of course, are the same, only changing the target calculus to the call-by-name standard one.

9. Conclusion and future work

This work is a case study of polarities in a fragment of an intuitionistic focused system (LJF) and reduction strategies in λ-calculus.

Related to the first subject, we have shown how to transform proofs having atoms assigned to different polarities inside the hereditarry Harrop fragment of LJF. Not surprisingly, this transformation process introduces cuts. In fact, it is related to the well known proof transformations from sequent calculus to natural deductions proof systems and vice versa.

However, the approach presented is really a novelty, since we have only one base system, changing the polarities on atoms.

We then showed the intention of applying these transformations in order to uniformly describe different evaluation strategies in λ-calculus. Although this part is still under construction, the idea is very simple: eliminating positive cuts introduces one smaller cut. This matches well the call-by-value behavior, where one has only one reduction – of a term to a value – then this value is passed to the sub terms of the term itself. On the other hand, eliminating negative cuts gives rise to several other cuts, which matches perfectly the idea of first passing the term to sub terms, then evaluating each one of them.

This work has intersection with several works. In particular, the idea of matching a certain focused logical system with a call-by-vaue lambda calculus is not new. In [3], Dyckhoff and Lengrand established a connection between LJQ and λ_C. For this, they presented an equational correspondence between these two calculi forming a bijection between the two sets of normal terms, and allowing reductions in each to be simulated by reductions in the other.

The nature of our approach is quite different. In fact, we rely in one logical system to describe both call-by-value an call-by-name reduction strategies. And we do not first propose cut reductions, then prove that they are admissible, but instead get the reductions for free based only on the basic logical system and focusing.

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