Focusing in linear meta-logic

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Abstract. It is well known how to use an intuitionistic meta-logic to specify natural deduction systems. It is also possible to use linear logic as a meta-logic for the specification of a variety of sequent calculus proof systems. Here, we show that if we adopt different *focusing* annotations for such linear logic specifications, a range of other proof systems can also be specified. In particular, we show that natural deduction (normal and non-normal), sequent proofs (with and without cut), tableaux, and proof systems using general elimination and general introduction rules can all be derived from essentially the same linear logic specification by altering focusing annotations. By using elementary linear logic equivalences and the completeness of focused proofs, we are able to derive new and modular proofs of the soundness and completeness of these various proofs systems for intuitionistic and classical logics.

1 Introduction

Logics and type systems have been exploited in recent years as frameworks for the specification of deduction in a number of logics. The most common such *meta-logics* and *logical frameworks* have been based on intuitionistic logic (see, for example, [3]) or dependent types (see [6, 16]). Such intuitionistic logics can be used to directly encode natural deduction style proof systems.

In a series of papers [9, 12, 11, 18, 19], Miller & Pimentel used classical linear logic as a meta-logic to specify and reason about a variety of sequent calculus proof systems. Since the encodings of such logical systems are natural and direct, the meta-theory of linear logic can be used to draw conclusions about the object-level proof systems. More specifically, in [11], a decision procedure was presented for determining if one encoded proof system is derivable from another. In the same paper, necessary conditions were presented (together with a decision procedure) for assuring that an encoded proof system satisfies cut-elimination. This last result used linear logic's dualities to formalize the fact that if the left and right introduction rules are suitable duals of each other then non-atomic cuts can be eliminated.

In this paper, we again use linear logic as a meta-logic but make critical use of the completeness of *focused proofs* for linear logic. Roughly speaking, focused proofs in linear logic divide sequent calculus proofs into two different phases: the *negative* phase involves rules that are invertible while the *positive* phase involves the focused non-invertible rules. In linear logic, it is clear to which phase each linear logic connective appears but it is completely arbitrary how atomic formulas can be assigned to these different phases. For example, all atomic formulas can be assigned a *negative polarity* or a *positive polarity* or, in fact, any mixture of these. The completeness of focused proofs then states that if a formula B is provable in linear logic and we fix on any polarity assignment to atomic formulas, then B will have a focused proof. (Soundness also holds.) Thus, while polarity assignment does not affect provability, it can result in strikingly different proofs. The earlier works of Miller & Pimentel assumed that all atoms were given negative polarity: this resulted in an encoding of object-level sequent calculus. As we shall show here, if we vary that polarity assignment, we can get other object-level proof systems represented. Thus, while provability is not affected, different, meta-level, focused proofs are built and these encode different object-level proof systems.

Our main contribution in this paper is illustrating how a range of proof systems can be seen as different focusing disciplines on the same or (meta-logically) equivalent sets of linear logic specifications. Soundness and relative completeness are generally trivial consequences of linear logic identities. In particular, we present examples based on sequent calculus and natural deduction [4], Generalized Elimination Rules [23], Free Deduction [15], the tableaux system KE [2], and Smullyan's Analytic Cut [22]. The adequacy of a given specification of inference rules requires first assigning polarity to meta-level atoms using in the specification: then adequacy is generally an immediate consequence of the focusing theorem of linear logic.

Finally, we attempt to point out how deep the equivalence of encoded proof systems goes by describing three levels of encoding adequacy: one where the provable set of formulas is the same, one where the completed proofs are in one-to-one correspondence, and one where (open) derivations (such as inference rules themselves) are also in one-to-one correspondence.

2 Preliminaries

2.1 Linear logic

We shall assume that the reader is familiar with linear logic. We review a few basic points here. *Literals* are either atomic formulas or their negations. We write $\neg F$ to denote the *negation normal form* of the formula F: that is, formulas computed by using de Morgan dualities and where negation has only atomic scope. The connectives $\otimes$ and $⅋$ and their units 1 and $\bot$ are *multiplicative*; the connectives $\oplus$ and $\&$ and their units 0 and $\top$ are *additive* connectives; $\forall$ and $\exists$ are (first-order) quantifiers; and $!$ and $?$ are the exponentials.

In general, we shall present *theories* in the linear meta-logic as appearing on the right-hand side of sequents. Thus, if $\mathcal{X}$ is a set of formulas (all the result of applying $?$ to existential closures), then we say that the formula B is derived using theory $\mathcal{X}$ if $\vdash B, \mathcal{X}$ is provable in linear logic. We shall also write $B \equiv C$ to denote the formula $(\neg B ⅋ C) \& (\neg C ⅋ B)$.

$(\Rightarrow_L)$	$\lfloor A \Rightarrow B \rfloor^\perp \otimes (\lceil A \rceil \otimes \lfloor B \rfloor)$	$(\Rightarrow_R)$	$\lceil A \Rightarrow B \rceil^\perp \otimes (\lfloor A \rfloor \invamp \lceil B \rceil)$
$(\wedge_L)$	$\lfloor A \wedge B \rfloor^\perp \otimes (\lfloor A \rfloor \oplus \lfloor B \rfloor)$	$(\wedge_R)$	$\lceil A \wedge B \rceil^\perp \otimes (\lceil A \rceil \,\&\, \lceil B \rceil)$
$(\vee_L)$	$\lfloor A \vee B \rfloor^\perp \otimes (\lfloor A \rfloor \,\&\, \lfloor B \rfloor)$	$(\vee_R)$	$\lceil A \vee B \rceil^\perp \otimes (\lceil A \rceil \oplus \lceil B \rceil)$
$(\forall_L)$	$\lfloor \forall B \rfloor^\perp \otimes \lfloor Bx \rfloor$	$(\forall_R)$	$\lceil \forall B \rceil^\perp \otimes \forall x \lceil Bx \rceil$
$(\exists_L)$	$\lfloor \exists B \rfloor^\perp \otimes \forall x \lfloor Bx \rfloor$	$(\exists_R)$	$\lceil \exists B \rceil^\perp \otimes \lceil Bx \rceil$
$(\perp_L)$	$\lfloor \perp \rfloor^\perp$	(t_R)	$\lceil t \rceil^\perp \otimes \top$

Fig. 1. The theory $\mathcal{L}$ used to encode various proof systems for minimal, intuitionistic, and classical logics.

(Id_1)	$\lfloor B \rfloor^\perp \otimes \lceil B \rceil^\perp$	(Id_2)	$\lfloor B \rfloor \otimes \lceil B \rceil$	(W_R)	$\lceil C \rceil^\perp \otimes \perp$
(Str_L)	$\lfloor B \rfloor^\perp \otimes ?\lfloor B \rfloor$	(Str_R)	$\lceil B \rceil^\perp \otimes ?\lceil B \rceil$		

Fig. 2. Specification of the identity rules (cut and initial) and of the structural rules (weakening and contraction).

2.2 Encoding object-logic formulas, sequents, and inference rules

We use linear logic as a meta-logic to encode object logics, in a similar fashion as done in [9, 19]. We shall assume that our meta-logic is a multi-sorted version of linear logic: in particular, the type o denotes meta-level formulas, the type *bool* denotes object-level formulas, and the type i will denote object-level terms. Object-level formulas are encoded in the usual way: in particular, the object-level quantifiers $\forall$, $\exists$ are given the type $(i \to bool) \to bool$ and the expressions $\forall(\lambda x.B)$ and $\exists(\lambda x.B)$ are written, respectively, as $\forall x.B$ and $\exists x.B$. To deal with quantified object-level formulas, our meta-logic will quantify over variables of types $i \to \cdots \to i \to bool$ (for 0 or more occurrences of i).

Encoding object-level sequents as meta-logic sequents is done by introducing two meta-level predicates of type $bool \to o$, written as $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$, and then writing the two-sided, object-level sequent $B_1, \ldots, B_n \vdash C_1, \ldots, C_m$ as the one-sided, meta-level sequent $\vdash \lfloor B_1 \rfloor, \ldots, \lfloor B_n \rfloor, \lceil C_1 \rceil, \ldots, \lceil C_m \rceil$. Thus formulas on the left of the object-level sequent are marked using $\lfloor\cdot\rfloor$ and formulas on the right of the object-level sequent are marked using $\lceil\cdot\rceil$. We shall assume that object-level sequents are pairs of either sets or multisets and that meta-level sequents are multisets of formulas. For convenience, if Γ is a (multi)set of formulas, $\lfloor \Gamma \rfloor$ (resp. $\lceil \Gamma \rceil$) denotes the multiset of atoms $\{\lfloor F \rfloor \mid F \in \Gamma\}$ (resp. $\{\lceil F \rceil \mid F \in \Gamma\}$).

Inference rules generally attribute to a logical connective two sets of "dual" inference rules: in sequent calculus, these correspond to the left-introduction and right-introduction rules while in natural deduction, these correspond to the introduction and elimination rules. Consider the linear logic formulas in Figure 1. When we display formulas in this manner, we intend that the named formula is actually the result of applying ? to existential closure of the formula. Thus, the formula named $\wedge_L$ is actually $?\exists A \exists B[\lfloor A \wedge B \rfloor^\perp \otimes (\lfloor A \rfloor \oplus \lceil B \rceil)]$. The formulas in Figure 1 help to provide the meaning of linear logic connectives in a rather abstract and succinct fashion. For example, the conjunction connective appears in two formulas: once in the scope of $\lfloor\cdot\rfloor$ and once in the scope of $\lceil\cdot\rceil$. Notice that there is no explicit reference to side formulas or any side conditions for any of these rules. We shall provide a much more in-depth analysis of the formulas in Figure 1 in the following sections.

The formulas in Figure 2 play a central role in this paper. The Id_1 and Id_2 formulas can prove the duality of the $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ predicates: in particular, one can prove in linear logic that

$$\vdash \forall B(\lceil B\rceil \equiv \lfloor B\rfloor^{\perp}) \mathbin{\&} \forall B(\lfloor B\rfloor \equiv \lceil B\rceil^{\perp}), Id_1, Id_2$$

Similarly, the formulas Str_L and Str_R allow us to prove the equivalences $\lfloor B\rfloor \equiv ?\lfloor B\rfloor$ and $\lceil B\rceil \equiv ?\lceil B\rceil$. The last two equivalences allows the weakening and contraction of formulas at both the meta-level and object-level. For instance, in the encoding of minimal logics, where structural rules are only allowed in the left-hand-side, one should include only the Str_L formula; while in the encoding of classical logics, where structural rules are allowed in both sides of a sequent, one should include both Str_L and Str_R formulas. Moreover, since the presence of these two formulas allows contracting and weakening of $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ atoms, one can show that the specification $\mathcal{L}\cup\{Str_L, Str_R\}$ is equivalent to the specification obtained from it but where the "additive rules" $\wedge_L, \wedge_R, \vee_L, \vee_R$ are replaced by the existential closure of their multiplicative versions, namely

$$\begin{array}{ll} \lceil A\wedge B\rceil^{\perp}\otimes(\lceil A\rceil\otimes\lceil B\rceil) & \lfloor A\wedge B\rfloor^{\perp}\otimes(\lfloor A\rfloor \parr \lfloor B\rfloor) \\ \lfloor A\vee B\rfloor^{\perp}\otimes(\lceil A\rceil\otimes\lceil B\rceil) & \lceil A\vee B\rceil^{\perp}\otimes(\lceil A\rceil \parr \lfloor B\rfloor). \end{array}$$

The formula W_R encodes the weakening right rule and is used to encode intuitionistic logics, where weakening, but not contraction, is allowed on formulas on the right-hand-side of a sequent.

2.3 Adequacy levels for encodings

When comparing deductive systems, one can easily identify several "levels of adequacy." Following Girard in [5, Chapter 7], we shall characterize our theorems as being from either Level 0, Level -1, or Level -2 (Girard considers also Level -3). Level 0 comparisons are made only by speaking of *provability*: a formula has a proof in one system if it has a proof in another system. Level -1 comparisons are made by comparing proofs object: the proofs of a given formula are in one-to-one correspondence with proofs in another system. If one uses the term "derivation" for incomplete proofs (proofs with open premises), then Level -2 comparison are made by comparing derivations (such as inference rules themselves): the derivations in one system are in one-to-one correspondence with those in another system. Standard completeness or relative completeness theorems are Level 0 theorems; full completeness results are Level -1 theorems. When we state equivalences between proof systems (usually between object-level proof systems and their meta-level encoding), we will often comment on which level the theorem should be placed.

2.4 A focusing proof system for linear logic

In [1], Andreoli proved the completeness of the focused proof system for linear logic given in Figure 3. Focusing proof systems involve applying inference rules in alternating *polarities*. In particular, formulas are *negative* if their top-level connective is either $\invamp, \bot, \&, \top, ?$, or $\forall$; formulas are *positive* if their top-level connective is $\oplus, 0, \otimes, 1, !$, or $\exists$. This polarity assignment is rather natural in the sense that all right introduction rules for negative formulas are invertible while such introduction rules for positive formulas are not necessarily invertible. The only formulas that are not given a polarities by the above assignment are the literals. Andreoli's completeness theorem can be interpreted as follows: If F is a provable linear logic formula, then for *any assignment* of polarities to the atomic formulas of linear logic, the sequent $\vdash \cdot : \cdot \Uparrow F$ is provable.

We point out two important aspects of this completeness theorem. First, the focus proof system only works on "annotated formulas" and not regular formulas. Here, the annotation is a mapping of atoms to polarities. (In intuitionistic and classical logics, one may also need to annotate conjunctions and disjunctions [8].) Notice that the rules $[I_1]$ and $[I_2]$ explicitly refer to the polarity assigned to literals. Second, an annotation does not affect *provability* but it may affect greatly the structure of (focused) proofs that are possible. In papers such as [8, 10], differences in annotations allowed one to build only top-down (goal-directed) proofs or only bottom-up (program-directed) proofs or combinations of both. In this paper, we shall illustrate how it is possible to use different polarity assignments (in the linear meta-logic) to derive different proof systems (of an object-logic). In particular, sequent calculus and natural deduction can be seen as *two different* annotations of the *same* linear logic specification of proof rules for (object-level) connectives.

Our linear meta-logic will yield specifications of object-logic proof systems only after we assign polarities to atoms of the form $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$: then our adequacy results will involve establishing relationships between focused meta-level proofs and object-level proof systems.

3 Sequent Calculus

We first consider how to encode sequent calculus systems for minimal, intuitionistic, and classical logics. The following three sets of formulas

$$\mathcal{L}_{lm} = (\mathcal{L} \setminus \{\bot_L, \Rightarrow_L\}) \cup \{Id_1, Id_2, Str_L, \Rightarrow'_L\} \qquad \mathcal{L}_{lj} = \mathcal{L}_{lm} \cup \{\bot_L, W_R\}$$
$$\mathcal{L}_{lk} = \mathcal{L} \cup \{Id_1, Id_2, Str_L, Str_R\}$$

where $\Rightarrow'_L$ is the formula $?\exists A \exists B[\lfloor A \Rightarrow B\rfloor^\perp \otimes (!\lceil A\rceil \otimes \lfloor B\rfloor)]$, are used to encode the LM, LJ and LK sequent calculus proof systems for minimal, intuitionistic, and classical logic (not displayed here to save space). These sets differ in the structural rules for $\lceil \cdot \rceil$, in the presence or absence of the formula $\bot_L$ and in the formula encoding the left introduction for implication: in the LM encoding, no structural rule is allowed in the right-hand-side formula; in the LJ encoding,

$$\frac{\vdash \Theta : \Gamma \Uparrow L}{\vdash \Theta : \Gamma \Uparrow L, \bot}\ [\bot] \qquad \frac{\vdash \Theta : \Gamma \Uparrow L, F, G}{\vdash \Theta : \Gamma \Uparrow L, F \parr G}\ [\parr] \qquad \frac{\vdash \Theta, F : \Gamma \Uparrow L}{\vdash \Theta : \Gamma \Uparrow L, ?F}\ [?]$$

$$\frac{}{\vdash \Theta : \Gamma \Uparrow L, \top}\ [\top] \qquad \frac{\vdash \Theta : \Gamma \Uparrow L, F \quad \vdash \Theta : \Gamma \Uparrow L, G}{\vdash \Theta : \Gamma \Uparrow L, F \,\&\, G}\ [\&] \qquad \frac{\vdash \Theta : \Gamma \Uparrow L, F[c/x]}{\vdash \Theta : \Gamma \Uparrow L, \forall x\, F}\ [\forall]$$

$$\frac{}{\vdash \Theta : \Downarrow 1}\ [1] \qquad \frac{\vdash \Theta : \Gamma \Downarrow F \quad \vdash \Theta : \Gamma' \Downarrow G}{\vdash \Theta : \Gamma, \Gamma' \Downarrow F \otimes G}\ [\otimes] \qquad \frac{\vdash \Theta : \Uparrow F}{\vdash \Theta : \Downarrow !\, F}\ [!]$$

$$\frac{\vdash \Theta : \Gamma \Downarrow F}{\vdash \Theta : \Gamma \Downarrow F \oplus G}\ [\oplus_l] \qquad \frac{\vdash \Theta : \Gamma \Downarrow G}{\vdash \Theta : \Gamma \Downarrow F \oplus G}\ [\oplus_r] \qquad \frac{\vdash \Theta, F : \Gamma \Downarrow F[t/x]}{\vdash \Theta : \Gamma \Downarrow \exists x\, F}\ [\exists]$$

$$\frac{}{\vdash \Theta : A_p^\bot \Downarrow A_p}\ [I_1] \qquad \frac{}{\vdash \Theta, A_p^\bot : \Downarrow A_p}\ [I_2] \qquad \frac{\vdash \Theta : \Gamma, S \Uparrow L}{\vdash \Theta : \Gamma \Uparrow L, S}\ [R \Uparrow]$$

$$\frac{\vdash \Theta : \Gamma \Downarrow P}{\vdash \Theta : \Gamma, P \Uparrow}\ [D_1] \qquad \frac{\vdash \Theta, P : \Gamma \Downarrow P}{\vdash \Theta, P : \Gamma \Uparrow}\ [D_2] \qquad \frac{\vdash \Theta : \Gamma \Uparrow N}{\vdash \Theta : \Gamma \Downarrow N}\ [R \Downarrow]$$

Fig. 3. The focused proof system for linear logic [1]. Here, L is a list of formulas, Θ is a multiset of formulas, Γ is a multiset of literals and positive formulas, A_p is a positive literal, N is a negative formula, P is not a negative literal, and S is a positive formula or a negated atom.

$$\dfrac{\dfrac{}{\vdash \mathcal{K} : \Downarrow \lfloor A \Rightarrow B \rfloor^\bot}\ [I_2] \qquad \dfrac{\dfrac{\vdash \mathcal{K} : \lceil A \rceil \Uparrow}{\vdash \mathcal{K} : \Downarrow !\lceil A \rceil}\ [!, R \Uparrow] \qquad \dfrac{\vdash \mathcal{K} : \lfloor B \rfloor, \lceil C \rceil \Uparrow}{\vdash \mathcal{K} : \lceil C \rceil \Downarrow \lfloor B \rfloor}\ [R \Downarrow, R \Uparrow]}{\vdash \mathcal{K} : \lceil C \rceil \Downarrow !\lceil A \rceil \otimes \lfloor B \rfloor}\ [\otimes]}{\dfrac{\vdash \mathcal{K} : \lceil C \rceil \Downarrow F}{\vdash \mathcal{K} : \lceil C \rceil \Uparrow \cdot}\ [D_2]}\ [2 \times \exists, \otimes]$$

Fig. 4. Here, the formula $A \Rightarrow B \in \Gamma$ and $\mathcal{K}$ denotes the set $\mathcal{L}_{lm}, \lfloor \Gamma \rfloor$.

the right-hand formula can be weakened; and in the LK encoding, contraction is also allowed (using the exponential ?). The $\bot_L$ formula only appears in the encodings of LJ and LK. In the theories for LM and LJ, the formula encoding the left introduction rule for implication contains a !. We will comment more about this difference later in this section.

If we fix the polarity of all meta-level atoms to be negative, then focused proofs using $\mathcal{L}_{lm}$, $\mathcal{L}_{lj}$, and $\mathcal{L}_{lk}$ yield encodings of the object-level proofs in LM, LJ, and LK. To illustrate why focusing is relevant, consider the encoding of the left introduction rule for $\Rightarrow$: selecting this rule at the object-level corresponds to focusing on the formula $F = \exists A \exists B[\lfloor A \Rightarrow B \rfloor^\bot \otimes (!\lceil A \rceil \otimes \lfloor B \rfloor)]$ (which is a member of $\mathcal{L}_{lm}$). The focused derivation in Figure 4 is then forced once F is selected for the focus: for example, the left-hand-side subproof must be an application of initial – nothing else will work with the focusing discipline. Notice that this meta-level derivation directly encodes the usual left introduction rule for $\Rightarrow$: the object-level sequents $\Gamma, B \vdash C$ and $\Gamma \vdash A$ yields $\Gamma, A \Rightarrow B \vdash C$.

Proposition 1. *Let $\Gamma \cup \Delta \cup \{C\}$ be a set of object-level formulas. Assume that all meta-level atomic formulas are given a negative polarity. Then*

1) $\Gamma \vdash_{lm} C$ *iff* $\vdash \mathcal{L}_{lm}, \lfloor\Gamma\rfloor : \lceil C\rceil \Uparrow$ 2) $\Gamma \vdash_{lj} C$ *iff* $\vdash \mathcal{L}_{lj}, \lfloor\Gamma\rfloor : \lceil C\rceil \Uparrow$
3) $\Gamma \vdash_{lk} \Delta$ *iff* $\vdash \mathcal{L}_{lk}, \lfloor\Gamma\rfloor, \lceil\Delta\rceil :\Uparrow$

This proposition is proved in [11, 19]. As stated, this proposition is a Level 0 result. It is easy to see that, for LM, LJ, and LK, a level -1 holds: that is, focusing proofs using $\mathcal{L}_{lm}$, $\mathcal{L}_{lj}$, or $\mathcal{L}_{lk}$ correspond directly to object-level sequent calculus proofs in LM, LJ, or LK, respectively. As is apparent from the example above concerning the left-introduction rule for $\Rightarrow$, we can actually get a Level -2 result: inference rules in the object-level sequents are in one-to-one correspondence with focused derivations in the meta-logic. To achieve the Level -2 result, the ! in the encoding of the implication left-introduction rule is important for minimal and intuitionistic logics.

If one removes the formula Id_2 from the sets $\mathcal{L}_{lm}$, $\mathcal{L}_{lj}$, and $\mathcal{L}_{lk}$, obtaining the sets $\mathcal{L}^f_{lm}$, $\mathcal{L}^f_{lj}$, and $\mathcal{L}^f_{lk}$, respectively, one can restrict the proofs encoded to cut free (object-level) proofs, represented by the judgments $\vdash^f_{lm}$ for minimal logic, $\vdash^f_{lj}$ for intuitionistic logic, and $\vdash^f_{lk}$ for classical logic.

Proposition 2. *Let $\Gamma \cup \Delta \cup \{C\}$ be a set of object-level formulas. Assume that all meta-level atomic formulas are given a negative polarity. Then*

1) $\Gamma \vdash^f_{lm} C$ *iff* $\vdash \mathcal{L}^f_{lm}, \lfloor\Gamma\rfloor : \lceil C\rceil \Uparrow$ 2) $\Gamma \vdash^f_{lj} C$ *iff* $\vdash \mathcal{L}^f_{lj}, \lfloor\Gamma\rfloor : \lceil C\rceil \Uparrow$
3) $\Gamma \vdash^f_{lk} \Delta$ *iff* $\vdash \mathcal{L}^f_{lk}, \lfloor\Gamma\rfloor, \lceil\Delta\rceil :\Uparrow$

As above, similar equivalences at Levels -1 and -2 can be proved.

4 Natural Deduction

The Figure 5 presents natural deduction using a sequent-style notation: sequents of the form $\Gamma \vdash_{nd} C \uparrow$, encoded as a meta-level sequent $\vdash \Sigma, \lfloor\Gamma\rfloor : \lceil C\rceil$ (for some multiset of formulas Σ), are obtained from the conclusion by a derivation (from bottom-up) where C is not the major premise of an elimination rule; and sequents of the form $\Gamma \vdash_{nd} C \downarrow$, encoded as a sequent $\vdash \Sigma, \lfloor\Gamma\rfloor : \lfloor C\rfloor^\perp$, are obtained from the set of hypotheses by a derivation (from top-down) where C is extracted from the major premise of an elimination rule. These two types of derivations meet either with a match rule M or with a switch rule S. These two types of sequents are used to distinguish general natural deduction proofs from the normal form proofs [20], where the switch rule is not allowed. We use the judgment $\vdash_{nm}$ to denote the existence of a natural deduction proof and the judgment $\vdash^n_{nm}$ to denote the existence of a normal natural deduction proof.

We can account for natural deduction in minimal logic by simply changing polarity assignment: in particular, atoms of the form $\lfloor\cdot\rfloor$ are now positive and all atoms of the form $\lceil\cdot\rceil$ have negative polarity. This change in polarity causes the

$$\frac{}{\Gamma, A \vdash_{nd} A \downarrow}\ [\text{Ax}] \quad \frac{\Gamma \vdash_{nd} F \uparrow \quad \Gamma \vdash_{nd} G \uparrow}{\Gamma \vdash_{nd} F \wedge G \uparrow}\ [\wedge I] \quad \frac{\Gamma \vdash_{nd} F \wedge G \downarrow}{\Gamma \vdash_{nd} F \downarrow}\ [\wedge E]$$

$$\frac{\Gamma \vdash_{nd} A_i \uparrow}{\Gamma \vdash_{nd} A_1 \vee A_2 \uparrow}\ [\vee I] \quad \frac{\Gamma \vdash_{nd} A \vee B \downarrow \quad \Gamma, A \vdash_{nd} C \uparrow (\downarrow)\ \Gamma, B \vdash_{nd} C \uparrow (\downarrow)}{\Gamma \vdash_{nd} C \uparrow (\downarrow)}\ [\vee E]$$

$$\frac{\Gamma, A \vdash_{nd} B \uparrow}{\Gamma \vdash_{nd} A \Rightarrow B \uparrow}\ [\Rightarrow I] \quad \frac{\Gamma \vdash_{nd} A \Rightarrow B \downarrow \quad \Gamma \vdash_{nd} A \uparrow}{\Gamma \vdash_{nd} B \downarrow}\ [\Rightarrow E] \quad \frac{}{\Gamma \vdash_{nd} t \uparrow}\ [tI]$$

$$\frac{\Gamma \vdash_{nd} A\{c/x\} \uparrow}{\Gamma \vdash_{nd} \forall x A \uparrow}\ [\forall I] \quad \frac{\Gamma \vdash_{nd} \forall x A \downarrow}{\Gamma \vdash_{nd} A\{t/x\} \downarrow}\ [\forall E] \quad \frac{\Gamma \vdash_{nd} A \downarrow}{\Gamma \vdash_{nd} A \uparrow}\ [\text{M}] \quad \frac{\Gamma \vdash_{nd} A \uparrow}{\Gamma \vdash_{nd} A \downarrow}\ [\text{S}]$$

$$\frac{\Gamma \vdash_{nd} \exists x A \downarrow \quad \Gamma, A\{a/x\} \vdash_{nd} C \uparrow (\downarrow)}{\Gamma \vdash_{nd} C \uparrow (\downarrow)}\ [\exists E] \quad \frac{\Gamma \vdash_{nd} A\{t/x\} \uparrow}{\Gamma \vdash_{nd} \exists x A \uparrow}\ [\exists I]$$

Fig. 5. Rules for minimal natural deduction - NM. In $[\vee L]$, $i \in \{1, 2\}$.

formula Id_2, which behaved like the cut rule in sequent calculus, to now behave like the switch rule, as illustrated by the following derivation.

$$\dfrac{\dfrac{}{\vdash \Sigma, \lfloor \Gamma \rfloor : \lfloor C \rfloor^{\perp} \Downarrow \lfloor C \rfloor}\ [I_1] \quad \dfrac{\vdash \Sigma, \lfloor \Gamma \rfloor : \lceil C \rceil \Uparrow}{\vdash \Sigma, \lfloor \Gamma \rfloor : \Downarrow \lceil C \rceil}\ [R \Downarrow, R \Uparrow]}{\dfrac{\vdash \Sigma, \lfloor \Gamma \rfloor : \lfloor C \rfloor^{\perp} \Downarrow \lfloor C \rfloor \otimes \lceil C \rceil}{\vdash \Sigma, \lfloor \Gamma \rfloor : \lfloor C \rfloor^{\perp} \Uparrow}\ [D_2, \exists]}\ [\otimes]$$

As the following proposition states, to obtain an encoding of normal form proofs, we do not include the formula Id_2.

Proposition 3. *Let $\Gamma \cup \{C\}$ be a set of object-level formulas and assume that all $\lceil \cdot \rceil$ atomic formulas are given a negative polarity and that all $\lfloor \cdot \rfloor$ atomic formulas are given a positive polarity. Then*

1) $\Gamma \vdash_{nm} C \uparrow$ *iff* $\vdash \mathcal{L}_{lm}, \lfloor \Gamma \rfloor : \lceil C \rceil \Uparrow$ 2) $\Gamma \vdash^{n}_{nm} C \uparrow$ *iff* $\vdash \mathcal{L}^{f}_{lm}, \lfloor \Gamma \rfloor : \lceil C \rceil \Uparrow$
3) $\Gamma \vdash^{n}_{nm} C \downarrow$ *iff* $\vdash \mathcal{L}^{f}_{lm}, \lfloor \Gamma \rfloor : \lfloor C \rfloor^{\perp} \Uparrow$

An equivalent Level -1 statement can also be proved.

Since the polarity assignment in a focused system does not affect provability, we obtain for free the following (Level 0) equivalences between LM and NM.

Corollary 1. *If $\Gamma \cup \{C\}$ be a set of object-level formulas, then*

$$\Gamma \vdash_{lm} C \textit{ iff } \Gamma \vdash_{nm} C \quad \textit{and} \quad \Gamma \vdash^{f}_{lm} C \textit{ iff } \Gamma \vdash^{n}_{nm} C.$$

Treating negation (in particular, falsity) in natural deduction presentations of intuitionistic and classical logics is not straightforward. We show in [14] that extra meta-logic formulas are needed to encode these systems. Since the treatment of negation in natural deduction is not one about focusing in the meta-level, we do not discuss this issue further here.

$$\frac{\Gamma \vdash_{ge} [A \vee B] \quad \Gamma, A \vdash_{ge} C \quad \Gamma, B \vdash_{ge} C}{\Gamma \vdash_{ge} C} \qquad \frac{\Gamma \vdash_{ge} [A \wedge B] \quad \Gamma, A, B \vdash_{ge} C}{\Gamma \vdash_{ge} C}$$

$$\frac{\Gamma \vdash_{ge} [A \Rightarrow B] \quad \Gamma \vdash_{ge} A \quad \Gamma, B \vdash_{ge} C}{\Gamma \vdash_{ge} C} \qquad \frac{\Gamma \vdash_{ge} [\forall x\, A] \quad \Gamma, A\{t/x\} \vdash_{ge} C}{\Gamma \vdash_{ge} C}$$

Fig. 6. Four general elimination rules. The major premise is marked with brackets.

5 Natural Deduction with General Elimination Rules

Schroeder-Heister proposed an extension of natural deduction in [21], which we call "general natural deduction", by using the general elimination rules, depicted in Figure 6, that treat all elimination rules as is usually done for disjunction elimination rule. To encode proofs in the general natural deduction, we assign negative polarity to $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ atoms, and use the set of formulas $\mathcal{L}_{ge}$, obtained from $\mathcal{L}_{lm}$ by removing the formulas $\vee_L, \wedge_L, \Rightarrow'_L, \forall_L$ and adding the existential closure of the following four formulas:

$$\begin{array}{ll} \lceil A \Rightarrow B\rceil \otimes (!\lceil A\rceil \otimes \lfloor B\rfloor) & \lceil \forall B\rceil \otimes \lfloor Bx\rfloor \\ \lceil A \vee B\rceil \otimes (\lfloor A\rfloor \,\&\, \lfloor B\rfloor) & \lceil A \wedge B\rceil \otimes (\lfloor A\rfloor \parr \lfloor B\rfloor) \end{array}$$

Proposition 4. *Let $\Gamma \cup \{C\}$ be a set of object-level formulas. Assume that all meta-level atomic formulas are given a negative polarity. Then $\Gamma \vdash_{ge} C$ iff $\vdash \mathcal{L}_{ge}, \lfloor\Gamma\rfloor : \lceil C\rceil \Uparrow$.*

An equivalent Level -1 statement can also be proved.

Corollary 2. *Let $\Gamma \cup \{C\}$ be a set of object-level formulas. Then $\Gamma \vdash_{ge} C$ iff $\Gamma \vdash_{lm} C$.*

Notice that there are two differences between the formulas displayed above and the original formulas in $\mathcal{L}_{lm}$ that they replace. 1) The presence of the multiplicative version of $\wedge_L$, and 2) the replacement of literals of the form $\lfloor B\rfloor^\perp$ by $\lceil B\rceil$. Moreover, notice that without the Id_2 formula the equivalence $\lfloor B\rfloor^\perp \equiv \lceil B\rceil$ is not satisfied and, therefore, the set of formulas in $\mathcal{L}_{ge}$ is not equivalent to $\mathcal{L}^f_{lm}$. Therefore, we relate general natural deduction to the formulation of LM that contains the cut rule.

Negri and Plato in [13] propose a different notion of normal proofs in general natural deduction: *Derivations in general normal form have all major premises of elimination rules as assumption.* In other words, the major premises, represented by the bracketed formula in the general elimination rules shown in Figure 6, are discharged assumptions. In our framework, this amounts to enforcing, by the use of polarity assignment to meta-level atoms, that the major premises are present in the set of assumptions. We use the set $\mathcal{L}^f_{lm}$ and assign negative polarity to all atoms of the form $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$, to encode general normal form proofs, represented by the judgment $\vdash^n_{ge}$.

Proposition 5. *Let $\Gamma \cup \{C\}$ be a set of object-level formulas. Assume that all meta-level atomic formulas are given a negative polarity. Then $\Gamma \vdash^n_{ge} C$ iff $\vdash \mathcal{L}^f_{lm}, \lfloor \Gamma \rfloor : \lceil C \rceil \Uparrow$.*

An equivalent Level -1 statement can also be proved.

It is easy to see in our framework that cut-free sequent calculus proofs can easily be obtained from general normal forms proofs, and vice-versa, since, to encode both systems, we use exactly the same formulas, $\mathcal{L}^f_{lm}$, and assign the same polarity to $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ atoms.

Corollary 3. *Let Γ be a set of formulas and let C be a formula. Then $\Gamma \vdash^n_{ge} C$ iff $\Gamma \vdash^f_{lm} C$.*

6 Free Deduction

In [15], Parigot introduced the *free deduction* proof system for classical logic that employed both the general elimination rules of the previous section and general introduction rules[1]. The general introduction rules are depicted in Figure 7.

$$\frac{\Gamma, A \vee B \vdash_{fd} \Delta \quad \Gamma \vdash_{fd} \Delta, A}{\Gamma \vdash_{fd} \Delta}\ [\vee GI] \qquad \frac{\Gamma, A \Rightarrow B \vdash_{fd} \Delta \quad \Gamma, A \vdash_{fd} \Delta, B}{\Gamma \vdash_{fd} \Delta}\ [\Rightarrow GI]$$

$$\frac{\Gamma, A \wedge B \vdash_{fd} \Delta \quad \Gamma \vdash_{fd} \Delta, A \quad \Gamma \vdash_{fd} \Delta, B}{\Gamma \vdash_{fd} \Delta}\ [\wedge GI]$$

$$\frac{\Gamma, \neg A \vdash_{fd} \Delta \quad \Gamma, A \vdash_{fd} \Delta}{\Gamma \vdash_{fd} \Delta}\ [\neg GI_1] \qquad \frac{\Gamma \vdash_{fd} \Delta, \neg A \quad \Gamma \vdash_{fd} \Delta, A}{\Gamma \vdash_{fd} \Delta}\ [\neg GI_2]$$

Fig. 7. The general introduction rules.

To encode free deduction proofs, we proceed similarly to the treatment of natural deduction with general eliminations rules. In particular, we replace in all formulas of $\mathcal{L}$, except the formula $\perp_L$, literals of the form $\lfloor B \rfloor^\perp$ by $\lceil B \rceil$ and literals of the form $\lceil B \rceil^\perp$ by $\lfloor B \rfloor$, and call the resulting set union $\{Id_1, Id_2, Str_L, Str_R\}$ as $\mathcal{L}_{fd}$. For example, the formula $\wedge_R$ in $\mathcal{L}$ is replaced by $?\exists A \exists B[\lfloor A \wedge B \rfloor \otimes (\lceil A \rceil \& \lceil B \rceil)]$ in $\mathcal{L}_{fd}$.

We assign negative polarity to the atoms $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ except the atom $\lfloor \perp \rfloor$, for which we assign positive polarity because of the different treatment of negation in free deduction.

Proposition 6. *Let $\Gamma \cup \Delta$ be a set of object-level formulas. Assume that all meta-level atomic formulas are given a negative polarity except the atom $\lfloor \perp \rfloor$, which is given positive polarity. Then $\Gamma \vdash_{fd} \Delta$ iff $\vdash \mathcal{L}_{fd}, \lfloor \Gamma \rfloor, \lceil \Delta \rceil :\Uparrow$.*

[1] It is interesting to note that later and independently, Negri and Plato also introduced general introduction rules in [13, p. 214].

An equivalent Level -2 statement can also be proved.

Since the encoding $\mathcal{L}_{fd}$ is logically equivalent to $\mathcal{L}_{lk}$, we can show that free deduction and LK are (Level 0) equivalent.

Corollary 4. *Let Γ and Δ be sets of formulas. Then $\Gamma \vdash_{fd} \Delta$ iff $\Gamma \vdash_{lk} \Delta$.*

Parigot notes that if one of the premises of the general rules is "killed", *i.e.*, it is always the conclusion of an initial rule, then one can obtain either sequent calculus or natural deduction proofs. The "killing" of a premise is accounted for in our framework by the use of polarities to enforce the presence of a formula in the context of the sequent. As done with the normal forms in general natural deduction, we can use the equivalences $\lfloor B \rfloor \equiv \lceil B \rceil^\perp$ and $\lfloor B \rfloor^\perp \equiv \lceil B \rceil$ and use either additive or multiplicative versions of the formulas in $\mathcal{L}$ to obtain from $\mathcal{L}_{fd}$ the equivalent sets $\mathcal{L}_{lk}$, which encodes LK, and the set $\mathcal{L}_{fd}^{nk}$ obtained from $\mathcal{L}_{lk}$ by removing the formulas $\Rightarrow_L, \vee_L, \wedge_L$ and adding the existential closure of the following three clauses:

$$\lceil A \Rightarrow B \rceil \otimes \left(\lceil A \rceil \otimes \lceil B \rceil^\perp\right) \qquad \lceil A \wedge B \rceil \otimes \left(\lceil A \rceil^\perp \oplus \lceil B \rceil^\perp\right)$$
$$\lceil A \vee B \rceil \otimes \left(\lceil A \rceil^\perp \otimes \lceil B \rceil^\perp\right).$$

The resulting set of formulas can be seen as an encoding of a multiple conclusion natural deduction proof system.

7 System KE

In the previous sections, we dealt with systems that contained rules with more premises than the corresponding rules in sequent calculus or natural deduction. Now, we move to the other direction and deal with systems that contain rules with fewer premises.

In [2], D'Agostino and Mondadori proposed the propositional tableaux system KE displayed in Figure 8. Here, the only rule that has more than one premise is the cut rule. In the original system, the cut inference rule appears with a side condition limiting cuts to be analytical cuts. Though it is possible to encode analytic cuts in our framework, as we show in [14], we consider here the more general form of cuts because it relates more directly to the other systems already presented.

To encode KE, we assign negative polarity to all atoms $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ and use the set of linear logic formulas, $\mathcal{L}_{ke}$, obtained from $\mathcal{L}_{lk}^p$ (the propositional fragment of $\mathcal{L}_{lk}$), by removing the formulas $\wedge_R, \Rightarrow_L, \vee_L, \vee_R, \perp_L$ and adding the existential closure of the following eight formulas:

$$\begin{array}{ll}
\lfloor A \Rightarrow B \rfloor^\perp \otimes (\lfloor A \rfloor^\perp \otimes \lfloor B \rfloor) & \lceil A \wedge B \rceil^\perp \otimes (\lfloor A \rfloor^\perp \otimes \lceil B \rceil) \\
\lfloor A \Rightarrow B \rfloor^\perp \otimes (\lceil A \rceil \otimes \lceil B \rceil^\perp) & \lceil A \wedge B \rceil^\perp \otimes (\lceil A \rceil \otimes \lfloor B \rfloor^\perp) \\
\lfloor A \vee B \rfloor^\perp \otimes (\lceil A \rceil^\perp \otimes \lfloor B \rfloor) & \lceil A \vee B \rceil^\perp \otimes (\lceil A \rceil \parr \lceil B \rceil) \\
\lfloor A \vee B \rfloor^\perp \otimes (\lfloor A \rfloor \otimes \lceil B \rceil^\perp) & \lceil \perp \rceil
\end{array}$$

$$\frac{\Gamma, A \vee B, B \vdash_{ke} A, \Delta}{\Gamma, A \vee B \vdash_{ke} A, \Delta}\ [\vee_{L1}] \quad \frac{\Gamma, A \vee B, A \vdash_{ke} B, \Delta}{\Gamma, A \vee B \vdash_{ke} B, \Delta}\ [\vee_{L2}] \quad \frac{\Gamma \vdash_{ke} A, B, A \vee B, \Delta}{\Gamma \vdash_{ke} A \vee B, \Delta}\ [\vee_{R}]$$

$$\frac{\Gamma, A \wedge B, A, B \vdash_{ke} \Delta}{\Gamma, A \wedge B \vdash_{ke} \Delta}\ [\wedge_{L}] \quad \frac{\Gamma, A \vdash_{ke} A \wedge B, B, \Delta}{\Gamma, A \vdash_{ke} A \wedge B, \Delta}\ [\wedge_{R1}] \quad \frac{\Gamma, B \vdash_{ke} A \wedge B, A, \Delta}{\Gamma, B \vdash_{ke} A \wedge B, \Delta}\ [\wedge_{R1}]$$

$$\frac{\Gamma, A, A \Rightarrow B, B \vdash_{ke} \Delta}{\Gamma, A, A \Rightarrow B \vdash_{ke} \Delta}\ [\Rightarrow_{L1}] \quad \frac{\Gamma, A \Rightarrow B \vdash_{ke} A, B, \Delta}{\Gamma, A \Rightarrow B \vdash_{ke} B, \Delta}\ [\Rightarrow_{L2}]$$

$$\frac{\Gamma, \neg A \vdash_{ke} A, \Delta}{\Gamma, \neg A \vdash_{ke} \Delta}\ [\neg_{L}] \quad \frac{\Gamma, A \vdash_{ke} \neg A, \Delta}{\Gamma \vdash_{ke} \neg A, \Delta}\ [\neg_{R}] \quad \frac{\Gamma, A \vdash_{ke} A \Rightarrow B, B, \Delta}{\Gamma \vdash_{ke} A \Rightarrow B, \Delta}\ [\Rightarrow_{R}]$$

$$\frac{}{\Gamma, A \vdash_{ke} A, \Delta}\ [Ax] \quad \frac{\Gamma, A \vdash_{ke} \Delta \quad \Gamma \vdash_{ke} A, \Delta}{\Gamma \vdash_{ke} \Delta}\ [Cut]$$

Fig. 8. The rules for the classical propositional logic KE.

Proposition 7. *Let $\Gamma \cup \Delta$ be a set of object-level formulas. Assume that all meta-level atomic formulas are given a negative polarity. Then $\Gamma \vdash_{ke} \Delta$ iff $\vdash \mathcal{L}_{ke}, \lfloor \Gamma \rfloor, \lceil \Delta \rceil :\Uparrow$.*

An equivalent Level -2 statement can also be proved.

The only differences between $\mathcal{L}_{lk}^{p}$ and $\mathcal{L}_{ke}$ are the use of multiplicative connectives instead of additive connectives, and that some atoms of the form $\lfloor \cdot \rfloor$ ($\lceil \cdot \rceil$) appear in the form $\lceil \cdot \rceil^{\perp}$ ($\lfloor \cdot \rfloor^{\perp}$). As before, we can show that the sets $\mathcal{L}_{lk}^{p}$ and $\mathcal{L}_{ke}$ are equivalent: the first difference is addressed by the presence of Str_L and Str_R and the second difference is addressed by the presence of Id_1 and Id_2.

Corollary 5. *Let Γ and Δ be a set of formulas. Then $\Gamma \vdash_{ke} \Delta$ iff $\Gamma \vdash_{lk}^{p} \Delta$, where $\vdash_{lk}^{p}$ is the judgment representing provability in the propositional fragment of LK.*

8 Smullyan's Analytic Cut System

To illustrate how one can capture another extreme in proof systems, we consider Smullyan's proof system for analytic cut (AC) [22], which is depicted in Figure 9. Here, all rules except the cut rule are axioms. As the name of the system suggests, Smullyan also assigned a side condition to the cut rule, allowing only analytical cuts. As in the previous section, we shall drop this restriction in order to make connections to previous systems easier (but we can account for it: see [14]).

We again assign negative polarity to $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ atoms and use the theory $\mathcal{L}_{ac}$ that results from collecting the formulas in $\{Id_1, Id_2, Str_L, Str_L\}$ with the formula $\lceil \perp \rceil$ and the existential closure of the following:

$$\begin{array}{ll}
\lfloor A \wedge B \rfloor^{\perp} \otimes \left(\lceil A \rceil^{\perp} \oplus \lceil B \rceil^{\perp} \right) & \lceil A \wedge B \rceil^{\perp} \otimes \left(\lfloor A \rfloor^{\perp} \otimes \lfloor B \rfloor^{\perp} \right) \\
\lfloor A \vee B \rfloor^{\perp} \otimes \left(\lceil A \rceil^{\perp} \otimes \lceil B \rceil^{\perp} \right) & \lceil A \vee B \rceil^{\perp} \otimes \left(\lfloor A \rfloor^{\perp} \oplus \lfloor B \rfloor^{\perp} \right) \\
\lfloor A \Rightarrow B \rfloor^{\perp} \otimes \left(\lfloor A \rfloor^{\perp} \otimes \lceil B \rceil^{\perp} \right) & \lceil A \Rightarrow B \rceil^{\perp} \otimes \left(\lceil A \rceil^{\perp} \oplus \lfloor B \rfloor^{\perp} \right)
\end{array}$$

$$\frac{}{\Gamma, A \vee B \vdash_{ac} A, B, \Delta} \, [\vee_L] \quad \frac{}{\Gamma, A \vdash_{ac} A \vee B, \Delta} \, [\vee_{R1}] \quad \frac{}{\Gamma, B \vdash_{ac} A \vee B, \Delta} \, [\vee_{R2}]$$

$$\frac{}{\Gamma, A \wedge B \vdash_{ac} A, \Delta} \, [\wedge_{L1}] \quad \frac{}{\Gamma, A \wedge B \vdash_{ac} B, \Delta} \, [\wedge_{L2}] \quad \frac{}{\Gamma, A, B \vdash_{ac} A \wedge B, \Delta} \, [\wedge_R]$$

$$\frac{}{\Gamma, A, A \Rightarrow B \vdash_{ac} B, \Delta} \, [\Rightarrow_L] \quad \frac{}{\Gamma \vdash_{ac} A, A \Rightarrow B, \Delta} \, [\Rightarrow_{R1}] \quad \frac{}{\Gamma, B \vdash_{ac} A \Rightarrow B, \Delta} \, [\Rightarrow_{R2}]$$

$$\frac{}{\Gamma, \neg A, A \vdash_{ac} \Delta} \, [\neg_L] \quad \frac{}{\Gamma \vdash_{ac} A, \neg A, \Delta} \, [\neg_R] \quad \frac{}{\Gamma, A \vdash_{ac} A, \Delta} \, [Ax]$$

$$\frac{\Gamma, A \vdash_{ac} \Delta \quad \Gamma \vdash_{ac} A, \Delta}{\Gamma \vdash_{ac} \Delta} \, [Cut]$$

Fig. 9. Smullyan's Analytic Cut System AC for classical propositional logic, except that the cut rule is not restricted.

Proposition 8. *Let $\Gamma \cup \Delta$ be a set of object-level formulas. Assume that all meta-level atomic formulas are given a negative polarity. Then $\Gamma \vdash_{ac} \Delta$ iff $\vdash \mathcal{L}_{ac}, \lfloor \Gamma \rfloor, \lceil \Delta \rceil :\Uparrow \cdot$.*

An equivalent Level -2 statement can also be proved.

The encoding above differs from $\mathcal{L}^p_{lk}$ as in ways similar to the differences between $\mathcal{L}^p_{lk}$ and $\mathcal{L}_{ke}$. By using the same reasoning as with the encoding $\mathcal{L}_{ke}$, we can show that AC is (Level 0) equivalent to the propositional fragment of LK.

Corollary 6. *Let Γ and Δ be a set of formulas. Then $\Gamma \vdash_{ac} \Delta$ iff $\Gamma \vdash^p_{lk} \Delta$, where $\vdash^p_{lk}$ is the judgment representing provability in the propositional fragment of LK.*

9 Related Work

A number of logical frameworks have been proposed to represent object-level proof systems. Many of these frameworks, as used in, for example, [3, 6, 16], are based on intuitionistic (minimal) logic principles. In such settings, the dualities that we employ here, for example, $\lfloor B \rfloor \equiv \lceil B \rceil^\perp$, are not available within the logic and this makes reasoning about Level 0 equivalence between object-level proof systems harder. Also, since minimal logic sequents must have a single conclusion, the storage of object-level formulas is generally done on the left-hand side of meta-level sequents (see [7, 17]) with some kind of "marker" for the right-hand side (such as the non-logical "refutation" marker # in [17]). The flexibility of having the four meta-level literals $\lfloor B \rfloor$, $\lceil B \rceil$, $\lfloor B \rfloor^\perp$, and $\lceil B \rceil^\perp$ is not generally available in such intuitionistic systems. While it is natural in classical linear logic to consider having some atoms assigned negative and some positive polarities, most intuitionistic systems consider only uniform assignments of polarities to meta-level atoms (usually negative in order to support goal-directed proof search): the ability to mix polarity assignments for different meta-level atoms can only be achieved in more indirect fashions in such settings.

The abstract logic programming presentation of linear logic called Forum [9] has been used to specify sequent calculus proof systems in a style similar to that used here. That presentation of linear logic was, however, also limited in that negation was not a primitive connective and that all atomic formulas were assumed to have negative polarity. The range of encodings contained in this paper are not directly available using Forum.

10 Conclusions and Further Remarks

We have shown that by employing different focusing annotations or using different sets of formulas that are (meta-logically) equivalent to $\mathcal{L}$, a range of sound and (relatively) complete object-level proof systems could be encoded. We have illustrated this principle by showing how linear logic focusing and logical equivalences can account for object-level proof systems based on sequent calculus, natural deduction, generalized introduction and elimination rules, free deduction, the tableaux system KE, and Smullyan's system employing only axioms and the cut rule.

Logical frameworks aim at allowing proof systems to be specified using compact and declarative specifications of inference rules. It now seems that a much broader range of possible proof systems can be further specified by allowing flexible assignment of polarity to meta-logical atoms (instead of making the usual assignment of some fixed, global polarity assignment). A natural next step would be to see what insights might be carried from this setting of linear-intuitionistic-classical logic to other, say, intermediate or sub-structural logics.

While focusing at the meta-level clearly provides a powerful normal form of proof, we have not described how to use the techniques presented in this paper to derive object-level focusing proof systems. Finding a means to derive such object-level normal form proofs is an interesting challenge that we plan to develop next.

Another interesting line of future research would be to consider differences in the *sizes* of proofs in these different paradigms since these differences can be related to the topic of comparing bottom-up and top-down deduction. Thus, it might be possible to flexibly change polarity assignments that would result in different and, hopefully, more compact presentations of proofs.

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