Multi-focused Proofs with Different Polarity Assignments

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Abstract

In this work, we will reason on how a given focused proof where atoms are assigned with some polarity can be transformed into another focused proof where the polarity assignment to atoms is changed. This will allow, in principle, transforming a proof obtained using one proof system into a proof using another proof system. More specifically, using the intuitionistic focused system LJF restricted to Harrop formulas, we define a procedure, introducing cuts, for transforming a focused proof where an atom is assigned with positive polarity into another focused proof where the same atom is assigned negative polarity and vice-versa. Then we show how to eliminate these cuts, obtaining a very interesting result: while the process of eliminating a cut on a positive atom gives rise to a proof with one smaller cut, in the negative case the number of introduced cuts grows exponentially. We end the paper by showing how to use maximal multi-focusing identify proofs in LJF, giving rise to a 1-1 translation between maximal proofs in LJF and proofs in the natural deduction system for intuitionistic logic NJ, restricted to Harrop formulas.

Keywords: Intuitionistic logic, Proof Systems, Focusing, Identity of proofs.

1 Introduction

In focused proof systems, such as Andreoli's original focused proof system [And92] for linear logic or Liang and Miller's LJF and LKF focused proof systems for intuitionistic and classical logics [LM09], connectives are classified as positive or negative, according to their right introduction rules: positive connectives have not necessarily invertible rules, while negative connectives are those whose right introduction rules are invertible. The polarity of a non atomic formula is then given

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by the polarity of its outermost connective. The interesting fact is that atomic formulas can be arbitrarily assigned as positive or negative, without affecting the completeness of the focusing discipline.

While this choice for the polarity of atomic formulas does not affect provability, it does affect the shape of the resulting focused proofs obtained. For instance, it has been shown that this choice can explain different proof search strategies, such as backward chaining and forward chaining [CPP08,LM09]. For another example, Nigam and Miller have shown in [NM10] that depending on the polarity assignments used for the atomic formula, one can, from the same logical theory, encode sequent calculus proofs or natural deduction ones.

In this paper, using the focused system LJF [LM09] for intuitionistic logic restricted to hereditarry Harrop formulas [MNPS91], we define a procedure, introducing cuts, for transforming a focused proof where an atom is assigned with positive polarity into another focused proof where the same atom is assigned negative polarity and vice-versa. We then show how to eliminate these cuts. Hence, we are able to transform a proof using a forward chaining strategy into a proof using backward chaining strategy or even obtain novel translations from sequent calculus to natural deduction and vice versa.

Interestingly, while the process of eliminating a cut on a positive atom gives rise to a proof with one smaller cut, in the negative case the number of introduced cuts grows exponentially. This difference in the cut-elimination algorithm is most definitely related to the different evaluation strategies according to the Curry-Howard isomorphism, where cut-elimination corresponds to computation in a functional programming setting. We plan to investigate this better in the future.

We also propose a new *multi-focused* system for intuitionistic logic, mLJF, and show how to identify proofs in this system modulo permutations. It turns out that these *maximal* proofs, when restricted to Harrop formulas, give some very interesting results: if atoms are restricted to the negative polarity, mLJF collapses to LJF, while if atoms are restricted to the positive polarity, for each provable sequent in LJF there is exactly one maximal proof. This means that a proof with negative atoms correspond to a proof with positive atoms and the correspondence is 1-1 up to permutation of rules. In this way we are able, for the first time, to give a correspondence between an intuitionistic focused system with positive atoms and Gentzen's natural deduction system NJ, solving completely the problem of identity of proofs in intuitionistic logic in the sequent calculus setting.

Finally, we sketch the dynamics of this correspondence in both sides, hence combining everything presented in the body of the paper.

The paper is organised as follows: Section 2 presents the system LJF and the logic programming fragment based on Harrop formulas, LJF_H; then Sections 3 and 4 show how to change polarities of atoms in LJF_H (introducing cuts) and how to eliminate cuts coming back to proofs in LJF_H; Section 5 presents the multi-focused system mLJF and the notion of maximal multi-focused proofs; Section 6 relates polarities with maximality in LJF_H and Section 7 concludes the paper and presents some ideas for continuing this work.

Negative Phase

$$\frac{}{\Gamma; \Theta, \mathit{false} \Rightarrow R}\ \mathit{false}_l \qquad \frac{\Gamma; \Theta \Rightarrow R}{\Gamma; \Theta, \mathit{true} \Rightarrow R}\ \mathit{true}_l \qquad \frac{\Gamma; \Theta, B, C \Rightarrow R}{\Gamma; \Theta, B \wedge^+ C \Rightarrow R}\ \wedge_l^+$$

$$\frac{\Gamma; \Theta, B \Rightarrow C}{\Gamma; \Theta \Rightarrow B \supset C}\ \supset_r \qquad \frac{\Gamma; \Theta, B \Rightarrow R \quad \Gamma; \Theta, C \Rightarrow R}{\Gamma; \Theta, B \vee C \Rightarrow R}\ \vee_l$$

$$\frac{\Gamma; \Theta \Rightarrow B \quad \Gamma; \Theta \Rightarrow C}{\Gamma; \Theta \Rightarrow B \wedge^- C}\ \wedge_r^- \qquad \frac{\Gamma; \Theta, B \Rightarrow R}{\Gamma; \Theta, \exists y B \Rightarrow R}\ \exists_l \qquad \frac{\Gamma; \Theta \Rightarrow B}{\Gamma; \Theta \Rightarrow \forall y B}\ \forall_r$$

Positive Phase

$$\frac{}{\Gamma; \cdot \rightarrow [\mathit{true}]}\ \mathit{true}_r \qquad \frac{\Gamma; \cdot \rightarrow [B] \quad \Gamma; \cdot \rightarrow [C]}{\Gamma; \cdot \rightarrow [B \wedge^+ C]}\ \wedge_r^+ \qquad \frac{\Gamma; \cdot \rightarrow [B] \quad \Gamma, [C]; \cdot \rightarrow P_a}{\Gamma, [B \supset C]; \cdot \rightarrow P_a}\ \supset_l$$

$$\frac{\Gamma, [B_i]; \cdot \rightarrow P_a}{\Gamma, [B_1 \wedge^- B_2]; \cdot \rightarrow P_a}\ \wedge_{li}^- \qquad \frac{\Gamma; \cdot \rightarrow [B_i]}{\Gamma; \cdot \rightarrow [B_1 \vee B_2]}\ \vee_{ri} \qquad \frac{\Gamma; \cdot \rightarrow [B[t/x]]}{\Gamma; \cdot \rightarrow [\exists x B]}\ \exists_r \qquad \frac{\Gamma, [B[t/x]]; \cdot \rightarrow P_a}{\Gamma, [\forall x B]; \cdot \rightarrow P_a}\ \forall_l$$

Structural Rules

$$\frac{N, \Gamma, [N]; \cdot \rightarrow P_a}{N, \Gamma; \cdot \Rightarrow P_a}\ D_l \qquad \frac{\Gamma; \cdot \rightarrow [P]}{\Gamma; \cdot \Rightarrow P}\ D_r \qquad \frac{\Gamma; P \Rightarrow \cdot P_a}{\Gamma, [P]; \cdot \rightarrow P_a}\ R_l \qquad \frac{\Gamma; \cdot \Rightarrow N}{\Gamma; \cdot \rightarrow [N]}\ R_r$$

$$\frac{\Gamma, \Omega; \Theta \Rightarrow R}{\Gamma; \Theta, \Omega \Rightarrow R}\ \mathtt{store} \qquad \frac{}{\Gamma, [A_n]; \cdot \rightarrow A_n}\ I_l \qquad \frac{}{\Gamma, A_p; \cdot \rightarrow [A_p]}\ I_r$$

Fig. 1. The LJF system. Here A_n denotes a negative atom, A_p a positive atom, P a positive formula, N a negative formula, P_a a positive formula or an atom and , Ω is a multiset of negative or atomic formulas. All other formulas are arbitrary and y is not free in Γ, Θ or R.

2 The focused proof system LJF for intuitionistic logic

There are a number os ways of defining a focused system from Gentzen's sequent system LJ for intuitionistic logic [Gir93,Her94,DJS95,DL06,LM07,LM09]. We choose the one first presented in [LM07], called LJF since it is the only one which allows positive and negative atoms in the same system.

In order to present the focused proof system LJF, we first classify the connectives $\wedge^+, \vee, \exists$, *true* and *false* as *positive* (their left introduction is necessarily invertible) and the connectives $\supset, \wedge^-$, and $\forall$ as *negative* (their right introduction rules are invertible). This dichotomy must also be extended to formulas. Concerning the atomic ones: some pre-chosen atoms are considered negative and the rest are considered positive. That is, one is free to assign as positive or negative the polarity to atoms. From this, a formula is positive if its main connective is positive or it is a *positive atom* and is negative if its main connective is negative or it is a *negative atom*.

The proof system LJF depicted in Figure 1 has four types of sequents.

(i) The sequent $\Gamma; \cdot \rightarrow [R]$ is a *right-focusing* sequent (the focus is R);

(ii) The sequent $\Gamma, [R]; \cdot \rightarrow P_a$: is a *left-focusing* sequent (with focus on R);

(iii) The sequent $\Gamma; \Theta \Rightarrow R$ is an *unfocused sequent*. Here, Γ contains negative formulas and positive atoms;

(iv) The sequent $\Gamma; \cdot \Rightarrow P_a$ is an instance of the previous sequent where Θ is empty and the formula in the succedent is positive or atomic.

As an inspection of the inference rules of LJF reveals, the search for a *focused* proof is composed of two alternating phases, and these phases are governed by polarities. The *negative phase* applies invertible (negative) rules until exhaustion: no backtracking during this phase of search is needed. The negative phase uses the third type of sequent above (the unfocused sequents): in that case, Θ contains positive or negative formulas, while R is either negative or positive. If Θ contains positive formulas, then an introduction rule (either $\wedge_l, \exists_l$, $true_l$, or $false_l$) is used to decompose it; negative formulas are moved to the Γ context (by using the `store` rule); if R is negative, the rules $\wedge^-, \supset_r$ are applied until R becomes positive or atomic. The end of the negative phase is represented by the fourth type of sequent. Such a sequent turns then to a focused one by using one of the decide rules, D_r or D_l. The application of one of these decide rules then selects a formula for focusing and switches proof search to the *positive phase* or *focused phase*. This focused phase then proceeds by applying sequences of inference rules on focused formulas: in general, backtracking may be necessary in this phase of search. The focusing phase ends with one of the *release rule* R_l or R_r.

As pointed out in [LM07], if all atoms are given negative polarity, the resulting proof system models backward chaining proof search and uniform proofs [MNPS91]. If positive atoms are permitted as well, then forward chaining steps can also be accommodated. Moreover, as in [NM10], it is possible in LJF to specify with the same intuitionistic theory sequent calculus proofs by using one polarity assignment and natural deduction proofs by using another polarity assignment.

Example 2.1 It is well known that the polarity assigned to atomic formulas does not change *provability*. On the other hand, the shape of proofs can differ a lot when different polarities are assigned to atoms. As an example, consider the Fibonacci program

$$\texttt{fib}(0,0) \wedge^+ \texttt{fib}(1,1) \wedge^+ \forall n,d,d'.[\texttt{fib}(n,d) \wedge^+ \texttt{fib}(n+1,d') \supset \texttt{fib}(n+2,d+d')]$$

Let $\Gamma = \texttt{fib}(0,0), \texttt{fib}(1,1), \forall n,d,d'.[\texttt{fib}(n,d) \wedge^+ \texttt{fib}(n+1,d') \supset \texttt{fib}(n+2,d+d')]$
If `fib` has negative bias, then *the only possible proof* of $\Gamma \longrightarrow \texttt{fib}(12,144)$ is

$$\dfrac{\dfrac{}{\Gamma,[\texttt{fib}(10+2,55+89)];\cdot \to \texttt{fib}(12,144)}\,(I_l) \quad \dfrac{\dfrac{\dfrac{\pi_1}{\Gamma;\cdot \Rightarrow \texttt{fib}(10,55)}}{\Gamma;\cdot \to [\texttt{fib}(10,55)]}\,R_r \quad \dfrac{\dfrac{\pi_2}{\Gamma;\cdot \Rightarrow \texttt{fib}(11,89)}}{\Gamma;\cdot \to [\texttt{fib}(11,89)]}\,R_r}{\Gamma;\cdot \to [\texttt{fib}(10,55) \wedge^+ \texttt{fib}(11,89)]}\,(\wedge_r^+)}{\dfrac{\Gamma,[\forall n,d,d'.[\texttt{fib}(n,d) \wedge^+ \texttt{fib}(n+1,d') \supset \texttt{fib}(n+2,d+d')]];\cdot \to \texttt{fib}(12,144)}{\Gamma;\cdot \Rightarrow \texttt{fib}(12,144)}\,(D_l)}\,(\forall_l,\supset_l)$$

where π_1 and π_2 continue following the backward chaining strategy. On the other hand, if `fib` is positive, the only possible way to start the proof is the following

$$\dfrac{\dfrac{\dfrac{\pi_3}{\Gamma,\texttt{fib}(0+2,0+1);\cdot \Rightarrow \texttt{fib}(12,144)}}{\Gamma,[\texttt{fib}(0+2,0+1)];\cdot \to \texttt{fib}(12,144)} \quad \dfrac{\dfrac{}{\Gamma;\cdot \to [\texttt{fib}(0,0)]}\,(I_r) \quad \dfrac{}{\Gamma;\cdot \to [\texttt{fib}(1,1)]}\,(I_r)}{\Gamma;\cdot \to [\texttt{fib}(0,0) \wedge^+ \texttt{fib}(1,1)]}\,(\wedge_r^+)}{\dfrac{\Gamma,[\forall n,d,d'.[\texttt{fib}(n,d) \wedge^+ \texttt{fib}(n+1,d') \supset \texttt{fib}(n+2,d+d')]];\cdot \to \texttt{fib}(12,144)}{\Gamma;\cdot \Rightarrow \texttt{fib}(12,144)}\,(D_l)}\,(\forall_l,\supset_l)$$

where π_3 can mix forward and backward chaining strategies. Note that the first derivation is exponential on size, while the smallest one in the second is linear.

The following result is trivially true, since right focused rules do not introduce left focused sequents.

Lemma 2.2 *Let Γ be a set of LJF-formulas. Let Ξ be a positive trunk, that is a derivation containing only rules from the positive phase, with end sequent of the form $\Gamma; \cdot \to [F]$, then there is no sequent focused on the left in Ξ.*

2.1 The logic programming fragment: LJF_H

In some parts of this paper (Sections 3, 4 and 6), we will restrict theories used to be the D-formulas and goals to be the G-formulas both specified by the grammar below:

$$G := A \mid G \wedge^+ G \mid D \supset G \mid \forall x\, G$$

$$D := A \mid G \supset A \mid \forall x.D$$

That is, we will only consider, in those Sections, sequents of the type $\mathscr{D} \vdash G$, where $\mathscr{D}$ is a set of D-formulas and G is a goal. This is a straightforward extension of the fragment of hereditary Harrop formulas used to describe uniform proofs [MNPS91], where A is an atomic formula. We will call the resulting system LJF_H.

We restrict our language to this fragment mainly for presentation reasons, as it considerably simplifies the machinery used in the following sections. In particular, it allows for a concise cut-elimination procedure involving only some cut permutations shown in Section 4, which will be used in the subsequent sections to demonstrate the connections of the polarity assignment to translation of proofs in different systems, as well as giving a hint on how the change of polarities gives rise to call-by-value and call-by-name reduction strategies.

3 Changing polarities

In this section, we show how to transform focused proof where an atom is assigned with one polarity to a focused proof where this same atom is assigned the opposite polarity. The transformations below might not preserve the size of a proof. In fact, it may well happen that after a proof is transformed from one proof system to another, the proof increases exponentially. Although this is relevant in some cases, such as in Proof Carrying Code, it is not that relevant when trying to unify the library of results obtained with different proof systems.

3.1 From positive to negative polarity

In this section we demonstrate how to transform a focused proof where an atom is assigned with positive polarity into another focused proof where the same atom is assigned negative polarity. Assume that Ξ is a proof where the atom A is assigned with positive polarity. We modify Ξ by induction from the leaves to the root on the number of reaction left and initial right rules applied on A. In particular, we perform the following operations:

The base case is when the proof ends with an initial right rule, which can only appear in positive derivations. We eliminate initial right rules by replacing the following subderivations appearing in a positive derivation:

$$\dfrac{}{\Gamma;\cdot \rightarrow [A]}\,I_r \quad\text{and}\quad \dfrac{\dfrac{}{\Gamma;\cdot\rightarrow[A]}\,I_r}{\Gamma;\cdot\Rightarrow A}\,D_r$$

by the following derivations, respectively:

$$\dfrac{\dfrac{\dfrac{}{\Gamma,[A];\cdot\rightarrow A}\,I_l}{\Gamma;\cdot\Rightarrow A}\,D_l}{\Gamma;\cdot\rightarrow[A]}\,R_r \quad\text{and}\quad \dfrac{\dfrac{}{\Gamma,[A];\cdot\rightarrow A}\,I_l}{\Gamma;\cdot\Rightarrow A}\,D_l$$

Notice that from the former derivations, it is the case that $A \in \Gamma$ and therefore we can, in the latter derivations, focus on A.

The other possible cases are when one of the rules $\supset_l$, $\wedge_l^-$ or $\forall_l$ are applied. In those cases, an instance of the cut rule is added. We illustrate the case of $\supset_l$, the others are similar and simpler.

$$\dfrac{\dfrac{\begin{matrix}\Xi_1\\ \Gamma;\cdot\rightarrow[G]\end{matrix} \quad \dfrac{\begin{matrix}\Xi_2\\ \Gamma,A;\cdot\Rightarrow G'\end{matrix}}{\Gamma,[A];\cdot\rightarrow G'}\,R_l,\texttt{store}}{\Gamma,[G\supset A];\cdot\rightarrow G'}\,\supset_l}{\Gamma;\cdot\Rightarrow G'}\,D_l \quad\Longrightarrow\quad \dfrac{\dfrac{\dfrac{\begin{matrix}\Xi_1'\\ \Gamma;\cdot\rightarrow[G]\end{matrix}\quad \dfrac{}{\Gamma,[A];\cdot\rightarrow A}\,I_l}{\Gamma,[G\supset A];\cdot\rightarrow A}\,\supset_l}{\Gamma;\cdot\Rightarrow A}\,D_l \quad \begin{matrix}\Xi_2'\\ \Gamma,A;\cdot\Rightarrow G'\end{matrix}}{\Gamma;\cdot\Rightarrow G'}\,cut$$

Here, the derivations Ξ_1' and Ξ_2' are obtained by applying the inductive hypothesis to Ξ_1 and Ξ_2 of smaller height and transforming all occurrences of A with positive polarity into negative polarity. Notice that, from Lemma 2.2, in the remaining of positive trunk in Ξ_1 there may not be any occurrences of reaction left rules, but only of initial right rules which are handled by the base case. Hence, this operation removes all reaction left rules over all the appearances of the atomic formula A.

Finally, after applying these operations, we obtain an LJF proof with cuts. To obtain a cut-free proof, we apply the cut-elimination theorem given in Section 4. The resulting proof is a cut-free focused proof where the polarity of the atom A is negative.

3.2 From negative to positive polarity

The idea to transform a proof where an atom A is assigned with negative polarity to a proof where the same atom appears with positive polarity is similar to the previous case. We perform the following operations to the original proof:

$$\dfrac{\dfrac{\dfrac{}{\Gamma,[A];\cdot\rightarrow A}\,I_l \quad \begin{matrix}\Xi\\ \Gamma;\cdot\rightarrow[G]\end{matrix}}{\Gamma,[G\supset A];\cdot\rightarrow A}\,\supset_l}{\Gamma;\cdot\Rightarrow A}\,D_l \quad\Longrightarrow\quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma,A;\cdot\rightarrow[A]}\,I_r}{\Gamma,A;\cdot\Rightarrow A}\,D_r}{\Gamma,[A];\cdot\rightarrow A}\,R_l \quad \begin{matrix}\Xi'\\ \Gamma;\cdot\rightarrow[G]\end{matrix}}{\Gamma,[G\supset A];\cdot\rightarrow A}\,\supset_l}{\Gamma;\cdot\Rightarrow A}\,D_l$$

To eliminate all occurrences of R_r, we will make use of the cut rule. Consider the following positive derivation containing R_r rules on the negative polarity atom A and whose last rule is D_r:

$$\dfrac{\dfrac{\overset{\Xi_1}{\Gamma;\cdot\rightarrow[G_1]} \quad \cdots \quad \dfrac{\overset{\Xi_i}{\Gamma;\cdot\Rightarrow A}}{\Gamma;\cdot\rightarrow[A]}\,R_r \quad \cdots \quad \overset{\Xi_n}{\Gamma;\cdot\rightarrow[G_n]}}{\Gamma;\cdot\rightarrow[G]}}{\Gamma;\cdot\Rightarrow G}$$

It can be transformed to the following derivation where A, where the number of reaction rules is reduced and this occurrence of A has positive polarity.

$$\dfrac{\overset{\Xi_i'}{\Gamma;\cdot\Rightarrow A} \qquad \dfrac{\dfrac{\overset{\Xi_1'}{\Gamma,A;\cdot\rightarrow[G_1]} \ \cdots \ \dfrac{}{\Gamma,A;\cdot\rightarrow[A]}\,I_r \ \cdots \ \overset{\Xi_n'}{\Gamma,A;\cdot\rightarrow[G_n]}}{\Gamma,A;\cdot\rightarrow[G]}}{\Gamma,A;\cdot\Rightarrow G}}{\Gamma;\cdot\Rightarrow G}\,cut$$

The proofs $\Xi_1',\ldots,\Xi_n'$ are obtained by applying the inductive hypothesis where A has positive polarity. The inductive hypothesis is applicable since their height are smaller and the number of reaction rules is decreased by at least one.

4 Cut-elimination

Instead of using the cut-elimination algorithm with several intra-phase cut-rules given in [LM09], we exploit the fact that the theories encoding proof systems are hereditary Harrop formulas to give a simpler cut-elimination procedure, with only inter-phase cut-rules.

4.1 If cut-formula is a positive atom

Our algorithm consists of basically two rewrite rules, depending on which decide rule is applied last on left premise of the cut rule. If it is D_r then it is necessarily the case that the atom A used in the cut is in the context Γ, which implies that the cut is not necessary:

$$\dfrac{\dfrac{\dfrac{}{\Gamma;\cdot\rightarrow[A]}\,I_r}{\Gamma;\cdot\Rightarrow A}\,D_r \qquad \overset{\Xi}{\Gamma,A;\cdot\Rightarrow G}}{\Gamma;\cdot\Rightarrow G}\,cut$$

This derivation reduces to the following derivation where the cut is eliminated:

$$\overset{\Xi}{\Gamma;\cdot\Rightarrow G}$$

For the second case, when the decide rule D_l is applied last in the left premise of the cut rule, we proceed as follows:

$$\dfrac{\dfrac{\dfrac{\Gamma_1;\cdot\to[B_1] \quad \cdots \quad \Gamma_n;\cdot\to[B_n] \quad \dfrac{\overset{\Xi_1}{\Gamma,A';A\Rightarrow}}{\Gamma,[A'];\cdot\to A}\,R_l,\mathtt{store}}{\Gamma,[F];\cdot\to A}}{\Gamma;\cdot\Rightarrow A}\,D_l \qquad \overset{\Xi_2}{\Gamma,A;\cdot\Rightarrow G}}{\Gamma;\cdot\Rightarrow G}\,cut$$

Since our theories are hereditary Harrop formulas, once the formula F is focused on, the resulting formula focused on the left is necessarily an atom. Moreover, the atom A' cannot be negative otherwise one would have to finish the proof with an I_l rule, but this is not possible since the atom appearing at the right-hand-side, A, is positive. Hence, it is necessarily the case that the atom A' is positive and since it is focused on the left, one releases focus.

We permute the atomic cut above the positive phase to the left as follows:

$$\dfrac{\dfrac{\Gamma_1;\cdot\to[B_1] \quad \cdots \quad \Gamma_n;\cdot\to[B_n] \quad \dfrac{\dfrac{\overset{\Xi_1}{\Gamma,A';\cdot\Rightarrow A} \quad \overset{\Xi_2}{\Gamma,A,A';\cdot\Rightarrow G}}{\Gamma,A';\cdot\Rightarrow G}\,cut}{\Gamma,[A'];\cdot\to G}\,R_l,\mathtt{store}}{\Gamma,[F];\cdot\to G}}{\Gamma;\cdot\Rightarrow G}\,D_l$$

Remark 4.1 Observe that the cut is replaced by another, appearing upper in the proof.

4.2 If cut-formula is a negative atom

It turns out that the cut may not permute upwards on the left premise if A is negative. In fact, on focusing on a left formula F like in the last Section, if the resulting atom focusing on the left is negative, it has necessarily to be A and the proof finishes with an I_l rule. For all other cases we could proceed like in the positive case.

There are two base cases:

$$\dfrac{\overset{\Xi}{\Gamma;\cdot\Rightarrow A} \quad \dfrac{\dfrac{}{\Gamma,A,[A];\cdot\to A}\,I_l}{\Gamma,A;\cdot\Rightarrow A}\,D_l}{\Gamma;\cdot\Rightarrow A}\,cut \qquad \Longrightarrow \qquad \overset{\Xi}{\Gamma;\cdot\Rightarrow A}$$

$$\dfrac{\overset{\Xi}{\Gamma;\cdot\Rightarrow A} \quad \dfrac{\dfrac{}{\Gamma,A,[A'];\cdot\to A'}\,I_l}{\Gamma,A;\cdot\Rightarrow A'}\,D_l}{\Gamma,A;\cdot\Rightarrow A'}\,cut \qquad \Longrightarrow \qquad \dfrac{\dfrac{}{\Gamma,[A'];\cdot\to A'}\,I_l}{\Gamma;\cdot\Rightarrow A'}\,D_l$$

The inductive cases are obtained by moving the cut rule upwards.

Let $\star$ be the maximum sequence of inference rules excluding decide rules appearing above the sequent $\Gamma,A;\cdot\Rightarrow G$ (hence $\star$ has only negative rules). Let n be the minimum length of the sub-derivations of $\star$. If $n>0$,

$$\dfrac{\stackrel{\Xi}{\Gamma; \cdot \Rightarrow A} \quad \dfrac{\stackrel{\Xi'}{\Gamma', A; \cdot \Rightarrow G'}}{\Gamma, A; \cdot \Rightarrow G}\ \star}{\Gamma; \cdot \Rightarrow G}\ cut$$

where $\Gamma \subseteq \Gamma'$.

If, on the other hand, $n = 0$, the last rule applied for proving $[\Gamma, A] \longrightarrow [G]$ is a decision rule. There are then two sub-cases: D_l and D_r.

In both cases, after finishing the focus phases (positive or negative) we will end up with a proof of the shape (ignoring the leaves):

$$\dfrac{\stackrel{\Xi}{\Gamma; \cdot \Rightarrow A} \quad \dfrac{\stackrel{\Xi_1}{\Gamma_1, A; \cdot \Rightarrow G_1} \quad \cdots \quad \stackrel{\Xi_n}{\Gamma_n, A; \cdot \Rightarrow G_n}}{\Gamma, A; \cdot \Rightarrow G}}{\Gamma; \cdot \Rightarrow G}\ cut$$

and the cut is moved upwards as follows:

$$\dfrac{\dfrac{\stackrel{\Xi}{\Gamma_1; \cdot \Rightarrow A} \quad \stackrel{\Xi_1}{\Gamma_1, A; \cdot \Rightarrow G_1}}{\Gamma_1; \cdot \Rightarrow G_1}\ cut \quad \cdots \quad \dfrac{\stackrel{\Xi}{\Gamma_n; \cdot \Rightarrow A} \quad \stackrel{\Xi_n}{\Gamma_n, A; \cdot \Rightarrow G_n}}{\Gamma_n; \cdot \Rightarrow G_n}\ cut}{\Gamma; \cdot \Rightarrow G}$$

Remark 4.2 Observe that, in this case, one cut is replaced by many others, and hence the size of proof grows exponentially.

5 Multi-focusing

It is well known [Her94,EDH15] that the negative fragment of sequent calculus corresponds to natural deduction proofs. For example, the sequent $a, a \supset b, b \supset c \vdash c$ has two different proofs in LJ:

$$\dfrac{\dfrac{}{a, b \supset c \vdash a}\ I \quad \dfrac{\dfrac{}{b \vdash b}\ I \quad \dfrac{}{c \vdash c}\ I}{b, b \supset c \vdash c}\ \supset}{a, a \supset b, b \supset c \vdash c}\ \supset_l \qquad \dfrac{\dfrac{\dfrac{}{a \vdash a}\ I \quad \dfrac{}{b \vdash b}\ I}{a, a \supset b \vdash b}\ \supset_l \quad \dfrac{}{c, a \supset b \vdash c}\ I}{a, a \supset b, b \supset c \vdash c}\ \supset_l$$

The first proof corresponds to forward and the second backward chaining. In LJF, if atoms are positive the only proof is the first one, while if they are negative, the only valid proof is the second.

In natural deduction, there is only one proof:

$$\dfrac{\dfrac{}{a, a \supset b, b \supset c \vdash b \supset c}\ I \quad \dfrac{\dfrac{}{a, a \supset b, b \supset c \vdash a \supset b}\ I \quad \dfrac{}{a, a \supset b, b \supset c \vdash a}\ I}{a, a \supset b, b \supset c \vdash b}\ \supset E}{a, a \supset b, b \supset c \vdash c}\ \supset E$$

which corresponds to the negative proof.

Example 5.1 Consider the sequent $\Gamma; \cdot \Rightarrow b \wedge^+ d$ where $\Gamma = \{a, c, a \supset b, c \supset d\}$). This sequent has 6 different proofs in LJF. If all atoms are negative, the only possible proof is

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, [a]; \cdot \to a}\, I_l}{\Gamma; \cdot \to [a]}\, R_r, D_l \qquad \dfrac{}{\Gamma, [b]; \cdot \to b}\, I}{\Gamma; \cdot \Rightarrow b}\, D_l, \supset_l \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, [c]; \cdot \to c}\, I_l}{\Gamma; \cdot \to [c]}\, D_l, \supset_l \qquad \dfrac{}{\Gamma, [d]; \cdot \to d}\, I_l}{\Gamma; \cdot \Rightarrow d}\, D_l, \supset_l}{\Gamma; \cdot \Rightarrow b \wedge^+ d}\, D_r, \wedge^+ R, R_r$$

But if atoms are positive, there are two possible proofs:

$$\dfrac{\dfrac{}{\Gamma; \cdot \to [a]}\, I_r \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, b; \cdot \to [c]}\, I_r \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, b, d; \cdot \to [b]}\, I_r \quad \dfrac{}{\Gamma, b, d; \cdot \to [d]}\, I_r}{\Gamma, b, d; \cdot \Rightarrow b \wedge^+ d}\, \wedge^+ R}{\Gamma, b, [d]; \cdot \to b \wedge^+ d}\, D_r, R_l}{\Gamma, b; \cdot \Rightarrow b \wedge^+ d}\, D_l, \supset_l}{\Gamma, [b]; \cdot \to b \wedge^+ d}\, R_l}{\Gamma; \cdot \Rightarrow b \wedge^+ d}\, D_l, \supset_l$$

and

$$\dfrac{\dfrac{}{\Gamma; \cdot \to [c]}\, I_r \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, d; \cdot \to [a]}\, I_r \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, b, d; \cdot \to [b]}\, I_r \quad \dfrac{}{\Gamma, b, d; \cdot \to [d]}\, I_r}{\Gamma, b, d; \cdot \Rightarrow b \wedge^+ d}\, \wedge^+ R}{\Gamma, d, [b]; \cdot \to b \wedge^+ d}\, D_r, R_l}{\Gamma, d; \cdot \Rightarrow b \wedge^+ d}\, D_l, \supset_l}{\Gamma, [d]; \cdot \to b \wedge^+ d}\, R_l}{\Gamma; \cdot \Rightarrow b \wedge^+ d}\, D_l, \supset_l$$

Observe that the proofs differ only in the order of the application of the implication.

We will show next how to use the maximal multi-focusing approach in order to identify proofs that differ only on the permutation of rules. We start by presenting mLJF, a multi-focused system for LJF.

The system mLJF has two kinds of formulas:

$$\begin{aligned} P, Q &:= A_p \mid \mathit{false} \mid \mathit{true} \mid P \wedge^+ Q \mid P \vee Q \mid \exists x.P(x) \mid \downarrow N \\ M, N &:= A_n \mid M \wedge^- N \mid P \supset N \mid \forall x.N(x) \mid \uparrow P \end{aligned}$$

where P, Q are positive while M, N are negative formulas. The symbols $\uparrow$ and $\downarrow$ mark the changing of polarities. The syntax for contexts is the following

$$\Delta := \cdot \mid \Delta, N \qquad \Gamma, \Omega := \Delta \mid p \qquad \Psi := [\Delta] \qquad \Theta := \cdot \mid \Theta, P - \{p\}$$

Finally, mLJF has three kinds of sequents:

- the sequent $\Gamma; \Theta \Rightarrow R$ is unfocused;

Positive Phase

$$\frac{}{\Gamma; \cdot \to [true]}\ true_r \quad \frac{\Gamma, \Psi; \cdot \to [B] \quad \Gamma, \Psi; \cdot \to [C]}{\Gamma, \Psi; \cdot \to [B \wedge^+ C]}\ \wedge_r^+ \quad \frac{\Gamma, \Psi, [B_i]; \cdot \to R}{\Gamma, \Psi, [B_1 \wedge^- B_2]; \cdot \to R}\ \wedge_{li}^-$$

$$\frac{\Gamma, \Psi_1; \cdot \to [B] \quad \Gamma, \Psi_2, [C]; \cdot \to R}{\Gamma, \Psi_1, \Psi_2, [B \supset C]; \cdot \to R}\ \supset_l \quad \frac{\Gamma, \Psi; \cdot \to [B_i]}{\Gamma, \Psi; \cdot \to [B_1 \vee B_2]}\ \vee_{ri}$$

$$\frac{\Gamma, \Psi; \cdot \to [B[t/x]]}{\Gamma, \Psi; \cdot \to [\exists x B]}\ \exists_r \quad \frac{\Gamma, \Psi, [B[t/x]]; \cdot \to R}{\Gamma, \Psi, [\forall x B]; \cdot \to R}\ \forall_l$$

Structural Rules

$$\frac{\Delta, \Gamma, [\Delta]; \cdot \to P_a}{\Delta, \Gamma; \cdot \Rightarrow P_a}\ mD_l \qquad \frac{\Delta, \Gamma, [\Delta]; \cdot \to [P]}{\Delta, \Gamma; \cdot \Rightarrow \uparrow P}\ mD_r$$

$$\frac{\Gamma; \Theta \Rightarrow P_a}{\Gamma, [\uparrow \Theta]; \cdot \to P_a}\ mR_l \qquad \frac{\Gamma; \Theta \Rightarrow N}{\Gamma, [\uparrow \Theta]; \cdot \to [\downarrow N]}\ mR_r$$

$$\frac{\Gamma, \Delta, \Omega; \Theta \Rightarrow R}{\Gamma; \Theta, \Omega, \downarrow \Delta \Rightarrow R}\ \mathtt{store} \qquad \frac{}{\Gamma, [A_n]; \cdot \to A_n}\ I_l \qquad \frac{}{\Gamma, A_p; \cdot \to [A_p]}\ I_r$$

Fig. 2. mLJF system. Here A_n, A_p, P and N are the same as in Figure 1, P_a represents either a formula of the kind $\uparrow P$ or an atomic formula and R is either P_a or a bracket formula. In mD_l, Δ is non empty.

- the sequent $\Gamma, \Psi; \cdot \longrightarrow R$ is focused on the left, where $\Psi \neq \emptyset$;
- the sequent $\Gamma, \Psi; \cdot \longrightarrow [R]$ is focused on the right (and possibly on the left).

The negative phase in mLJF is the same as in LJF. The rest of the rules for mLJF are similar to the ones presented in Figure 1, only now considering possibly multi-focused contexts (Figure 2). Note that we can unfocus if and only if *every* focused formula is marked with arrows.

The following theorem is straightforward: just note that if we erase the $\uparrow$ and $\downarrow$ arrows and the context Ψ, and if we restrict Δ to a singleton in mD_l and to the empty set in mD_r, mLJF collapses to LJF.

Theorem 5.2 *mLJF is correct and complete with respect to LJF.*

Observe that the rule $\supset_l$ has a "linear" flavour as the focused left context splits on the premise sequents. This is only an operational trick in order to make maxi-multi-focalization possible.

Example 5.3 If restricted to positive atoms, there are now four proofs of the sequent presented in Example 5.1: focusing on $a \supset\uparrow b$ first, focusing on $c \supset\uparrow d$ first, or focusing on both at the same time and then applying the implication rules in the two possible orders. These two last proofs collapse to one if we consder the equivalent class of proofs modulo permutation of rules

$$\dfrac{\dfrac{}{\Gamma; \cdot \to [a]}\ I_r \quad \dfrac{}{\Gamma; \cdot \to [c]}\ I_r \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, b, d; \cdot \to [b]}\ I_r \quad \dfrac{}{\Gamma, b, d; \cdot \to [d]}\ I_r}{\Gamma, b, d; \cdot \to [b \wedge^+ d]}\ \wedge^+ R}{\Gamma, b, d; \cdot \Rightarrow b \wedge^+ d}\ mR_r}{\Gamma, [\uparrow b, \uparrow d]; \cdot \to b \wedge^+ d}\ mR_l, \mathtt{store}}{\dfrac{\Gamma, [a \supset\uparrow b, c \supset\uparrow d]; \cdot \to b \wedge^+ d}{\Gamma; \cdot \Rightarrow b \wedge^+ d}\ mD_l}\ 2 \times (\supset_l)$$

In this case, we say that the application of mD_l rule is *maximal*, that is, if it chooses the maximal possible set Δ for focusing. And it gives rise to a *synthetic connective* [MP13], that is, a connective that combines the application of various rules in one. Finally, observe that this maximal proof is possible only due to the splitting of the left focused context in the rule $\supset_l$, since the application of I_r on proving a and c implies that we cannot have any other focused formulas.

5.1 Maximal multi-focusing

We will now formalise the notion of *maximal multi-focusing* and *equivalence of proofs*, presented intuitively in the last example.

The following definitions are adaptations of the ones in [CMS08,CHM12] to mLJF:

Definition 5.4 The proofs Ξ_1 and Ξ_2 of the same mLJF sequent are *locally permutatively equivalent*, written $\Xi_1 \sim \Xi_2$, if each can be rewritten to the other using local permutations. Ξ_1 and Ξ_2 are *permutatively equivalent*, written $\Xi_1 \approx \Xi_2$, if they are locally permutatively equivalent and each can be rewritten to the other using permutations.

For example,

$$\dfrac{\dfrac{\overset{\Xi}{\Gamma; \Theta, B, C, D \Rightarrow E}}{\Gamma; \Theta, B \wedge^+ C, D \Rightarrow E}\ \wedge_l^+}{\Gamma; \Theta, B \wedge^+ C \Rightarrow D \supset E}\ \supset_r \quad \sim \quad \dfrac{\dfrac{\overset{\Xi}{\Gamma; \Theta, B, C, D \Rightarrow E}}{\Gamma; \Theta, B, C \Rightarrow D \supset E}\ \supset_r}{\Gamma; \Theta, B \wedge^+ C \Rightarrow D \supset E}\ \wedge_l^+$$

In fact, since all negative rules are invertible, they are permutable. This means that the whole negative phase collapse to one step, modulo permutations.

In the positive phase the permutability of rules depends on the polarities of formulas. We will come back to this later.

Non-locally permutatively equivalent proofs, on the other hand, require considering permutations of entire phases. As in [And01,CMS08], we call a neighbouring pair of phases, with the bottom phase positive and the top phase negative, a bipole. Consider two neighbouring bipoles: if the positive phase of the top bipole permutes with the negative phase of the bottom bipole, then in an unfocused form we can perform the permutation and merge the two bipoles by uniting their positive and negative phases, obtaining another (multi-)focused proof. This operation obviously terminates, giving rise to the following definition and theorem.

Definition 5.5 If a proof Ξ in mLJF ends with an instance of mD_l or mD_r, let $\texttt{foci}(\Xi)$ is defined as the multiset of foci in the premise of that instance. We say that this instance of mD_l or mD_r is *maximal* if and only if, for every $\Xi' \approx \Xi$, $\texttt{foci}(\Xi') \subseteq \texttt{foci}(\Xi)$. A proof in *mLJF* is maximal if and only if every instance of mD_l or mD_r in it is maximal.

Theorem 5.6 *Every sequent provable in mLJF has a maximal proof.*

The proofs presented in Example 5.3 are maximal, while the last two ones in Example 5.1 are not. But they can be transformed, via non-local permutations, to the ones in Example 5.3.

6 Maximal multi-focusing and Harrop formulas

The restriction of mLJF to Harrop formulas (here called mLJF_H) gives very interesting results.

Theorem 6.1 *If all atoms are negative then $mLJF_H = LJF_H$, that is, when restricted to Harrop formulas, multi-focused proofs are the same as singly focused proofs if only negative atoms are considered.*

Proof. Consider the proof

$$\dfrac{\dfrac{\overset{\Xi_1}{\Gamma, G \supset A, \Psi_1; \cdot \to [G]} \quad \overset{\Xi_2}{\Gamma, G \supset A, \Psi_2, [A]; \cdot \to C}}{\Gamma, G \supset A, \Psi_1, \Psi_2, [G \supset A]; \cdot \to C} \supset_l}{\Gamma, G \supset A; \cdot \Rightarrow C} \, mD_l$$

If A is a negative atom, Ξ_2 must be the application of the initial axiom I_l and hence $A = C$ and $\Psi_2 = \emptyset$. Now, it should be the case that $\Psi_1 = \emptyset$. If not, observe that it cannot exist a negative atom $n \in \Psi_1$, since G is focused on the right (and focused negative atoms should finish the proof). Hence either there exists $G' \supset A'$ or $\forall x.D$ in Ψ_1. But applying $\supset_l$ in a sequent of the type $\Gamma, G \supset A, \Psi_1; \cdot \to [G]$ will produce a sequent of the form $\Gamma, G \supset A, \Psi_1', [A']; \cdot \to [G]$, which is forbidden since A' is atomic negative (hence there can be no focused formula on the right of the sequent). On the other hand, applying $\forall_l$ will substitute a focused formula $\forall x.D$ by the focused formula D; in this case, the focused context on the left will always produce another one, and the result follows by induction.

That is, there are not non-local permutations, foci in maximal multi-focused formulas have exactly one element, hence $\text{mLJF}_H = \text{LJF}_H$. The other cases are similar and simpler. □

Corollary 6.2 *If all atomic formulas are negative, any provable sequent in $mLJF_H$ has only one possible proof.*

In the positive case we also have a fascinating result.

Theorem 6.3 *For each provable sequent in $mLJF_H$, if all atoms are positive then there is only one maximal proof for it. That is, when restricted to Harrop formulas with only positive atoms, multi-focused proofs can be equated to one maximally focused proof.*

Proof. Consider the maximal proof Ξ

$$\dfrac{\dfrac{\overset{\Xi_1}{\Gamma, G \supset\uparrow A, \Psi_1; \cdot \to [G]} \quad \overset{\Xi_2}{\Gamma, G \supset\uparrow A, \Psi_2, [\uparrow A]; \cdot \to C}}{\Gamma, G \supset\uparrow A, \Psi_1, \Psi_2, [G \supset\uparrow A]; \cdot \to C} \supset_l}{\Gamma, G \supset\uparrow A; \cdot \Rightarrow C} \, mD_l$$

If G is a purely positive formula, Ψ_1 should be empty and there are no rules up to permute with the rightmost premise. If $G = \downarrow N$, a number of things can happen: if Ψ_1 is a (possibly empty) set of the form $\uparrow \Delta$, then focus will be lost and there will be a change of phases. Since Ξ is maximal, there is no way of permuting these phases. If Ξ_1 ends with $\supset_l$ or $\forall_l$, then these rules are locally permutable with $\supset_l$. For example, if $G' \supset\uparrow A' \in \Psi_1$ then

$$\dfrac{\dfrac{\dfrac{\overset{\Xi_1'}{\Gamma, G \supset\uparrow A, \Psi_1'; \cdot \to [G']} \quad \overset{\Xi_1''}{\Gamma, G \supset\uparrow A, \Psi_2', [\uparrow A']; \cdot \to [G]}}{\Gamma, G \supset\uparrow A, \Psi_1; \cdot \to [G]}\supset_l \quad \overset{\Xi_2}{\Gamma, G \supset\uparrow A, \Psi_2, [\uparrow A]; \cdot \to C}}{\Gamma, G \supset\uparrow A, \Psi_1, \Psi_2, [G \supset\uparrow A]; \cdot \to C}\supset_l}{\Gamma, G \supset\uparrow A; \cdot \Rightarrow C}\, mD_l$$

is locally equivalent to

$$\dfrac{\dfrac{\overset{\Xi_1'}{\Gamma, G \supset\uparrow A, \Psi_1'; \cdot \to [G']} \quad \dfrac{\overset{\Xi_1''}{\Gamma, G \supset\uparrow A, \Psi_2', [\uparrow A']; \cdot \to [G]} \quad \overset{\Xi_2}{\Gamma, G \supset\uparrow A, \Psi_2, [\uparrow A]; \cdot \to C}}{\Gamma, G \supset\uparrow A, \Psi_2, [G \supset\uparrow A, \uparrow A']; \cdot \to C}\supset_l}{\Gamma, G \supset\uparrow A, \Psi_1', \Psi_2, [G \supset\uparrow A, G' \supset\uparrow A']; \cdot \to C}\supset_l}{\Gamma, G \supset\uparrow A; \cdot \Rightarrow C}\, mD_l$$

The analysis is similar and simpler for Ψ_2 or in the case that multi-focusing is also on the right (mD_r). □

Corollary 6.4 *There is a 1-1 correspondence between maximal proofs in* $mLJF_H$ *restricted to positive atoms and proofs in* $mLJF_H$ *restricted to negative atoms. Hence there is a 1-1 correspondence between* $mLJF_H$ *restricted to positive atoms and proofs in NJ restricted to Harrop formulas.*

We will finish this section by sketching how these correspondences work, using the process developed in Sections 3 and 4.

From positive to negative. The process of changing polarities of atoms will transform a cut-free proof in mLJF$_H$ into a proof with cuts.

$$\dfrac{\dfrac{\overset{\Xi_1}{\Gamma; \cdot \to [G]} \quad \dfrac{\overset{\Xi_2}{\Gamma, A; \cdot \Rightarrow G'}}{\Gamma, [A]; \cdot \to G'}R_l, \texttt{store}}{\Gamma, [G \supset A]; \cdot \to G'}\supset_l}{\Gamma; \cdot \Rightarrow G'}D_l \quad \Longrightarrow \quad \dfrac{\dfrac{\dfrac{\overset{\Xi_1'}{\Gamma; \cdot \to [G]} \quad \dfrac{}{\Gamma, [A]; \cdot \to A}I_l}{\Gamma, [G \supset A]; \cdot \to A}\supset_l}{\Gamma; \cdot \Rightarrow A}D_l \quad \overset{\Xi_2'}{\Gamma, A; \cdot \Rightarrow G'}}{\Gamma; \cdot \Rightarrow G'}\, cut$$

We will denote by Ξ the leftmost subproof above the cut.

The cut-elimination process on negative atoms will (i) permute down the focused rule on the right premise above the cut (if any) and (ii) add a higher cut to every possible top premise appearing when the focused phase is over [4].

$$\dfrac{\dfrac{\overset{\Xi}{\Gamma_1; \cdot \Rightarrow A} \quad \overset{\Xi_1}{\Gamma_1, A; \cdot \Rightarrow G_1}}{\Gamma_1; \cdot \Rightarrow G_1}\, cut \quad \ldots \quad \dfrac{\overset{\Xi}{\Gamma_n; \cdot \Rightarrow A} \quad \overset{\Xi_n}{\Gamma_n, A; \cdot \Rightarrow G_n}}{\Gamma_n; \cdot \Rightarrow G_n}\, cut}{\Gamma; \cdot \Rightarrow G'}$$

Consider the proof

$$\dfrac{\overset{\Xi}{\Gamma_i; \cdot \Rightarrow A} \quad \overset{\Xi_i}{\Gamma_i, A; \cdot \Rightarrow G_i}}{\Gamma_i; \cdot \Rightarrow G_i}\, cut$$

[4] Here we abuse the notation and use Ξ also for its weakened version, substituting Γ by Γ_i, where $\Gamma \subseteq \Gamma_i$.

If the last rule of Ξ_i is the identity on A, then $G_i = A$ and hence the proof above is substituted by Ξ. If the last rule of Ξ_i is the identity on a formula other than A, then the cut is eliminated. Finally, if the last rule of Ξ_i is not the identity, we continue moving the cut up, together with Ξ. This will eliminate all the uppermost cuts and completely determine the order of application of rules in the negative case.

As an example, if we take either of the last two proofs in Example 5.1, this process will give the first proof, where the conjunction moves down and the implications occur in parallel branches of the proof.

From negative to positive. The proof

$$\dfrac{\dfrac{\dfrac{\Xi_1}{\Gamma;\cdot\rightarrow[G_1]} \quad \cdots \quad \dfrac{\dfrac{\Xi_i}{\Gamma;\cdot\Rightarrow A}}{\Gamma;\cdot\rightarrow[A]}\,R_r \quad \cdots \quad \dfrac{\Xi_n}{\Gamma;\cdot\rightarrow[G_n]}}{\Gamma;\cdot\rightarrow[G]}}{\Gamma;\cdot\Rightarrow G}$$

is transformed into

$$\dfrac{\dfrac{\Xi_i'}{\Gamma;\cdot\Rightarrow A} \qquad \dfrac{\dfrac{\dfrac{\Xi_1'}{\Gamma,A;\cdot\rightarrow[G_1]} \quad \cdots \quad \dfrac{}{\Gamma,A;\cdot\rightarrow[A]}\,I_r \quad \cdots \quad \dfrac{\Xi_n'}{\Gamma,A;\cdot\rightarrow[G_n]}}{\Gamma,A;\cdot\rightarrow[G]}}{\Gamma,A;\cdot\Rightarrow G}}{\Gamma;\cdot\Rightarrow G}\,cut$$

We will call Π_1 the rightmost subproof above the *cut*. Now if Ξ_i' has the form

$$\dfrac{\dfrac{\Gamma_1;\cdot\rightarrow[B_1] \quad \cdots \quad \Gamma_n;\cdot\rightarrow[B_n] \quad \dfrac{\dfrac{\Pi_1}{\Gamma,A';\cdot\Rightarrow A}}{\Gamma,[A'];\cdot\rightarrow A}\,R_l,\texttt{store}}{\Gamma,[F];\cdot\rightarrow A}}{\Gamma;\cdot\Rightarrow A}\,D_l$$

we can move the cut up

$$\dfrac{\dfrac{\Gamma_1;\cdot\rightarrow[B_1] \quad \cdots \quad \Gamma_n;\cdot\rightarrow[B_n] \qquad \dfrac{\dfrac{\dfrac{\Pi_2}{\Gamma,A';\cdot\Rightarrow A} \quad \dfrac{\Pi_1}{\Gamma,A,A';\cdot\Rightarrow G}}{\Gamma,A';\cdot\Rightarrow G}\,cut}{\Gamma,[A'];\cdot\rightarrow G}\,R_l,\texttt{store}}{\Gamma,[F];\cdot\rightarrow G}}{\Gamma;\cdot\Rightarrow G}\,D_l$$

Observe that focusing on the right is eliminated and, depending on the choice of A in Ξ_i, we may have different but permutatively equivalent proofs. In Example 5.1, starting from the first proof, we get the second proof if $A = a$ and the third if $A = c$.

7 Conclusion and future work

In this work, we have proposed a multi-focused system mLJF for the focused intuitionistic system LJF [LM07]. We then showed how to use the notion of maximal proofs in order to identify proofs in intuitionistic logic. The same results have been established in [CMS08] for the multiplicative-aditive fragment of linear logic and in [CHM12] for classical logic.

This is an important step towards solving the problem of identity of proofs in intuitionistic logic in the sequent calculus setting. In fact, when restricted to Harrop formulas, we have completely solved the problem (see Theorems 6.1 and 6.3). We hope to be able to expand these results for the whole intuitionistic logic.

But a very nice line of research to pursue is to relate the procedure given in Sections 3 and 4 in order to relate call-by-name and call-by-value. In particular, as noted in Remarks 4.1 and 4.2, systems restricted to positive atoms have a call-by-value behavior, where one cut is substituted by another on eliminating the cut. This has the flavour of linear reduction steps, evaluating the argument first for then passing it as a parameter. On the other hand, systems restricted to negative atoms have a call-by-name behavior, where one cut is substituted by possible many others, capturing well the notion of first passing the argument, then reducing all possible occurrences of it in the term.

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