Exploiting non-canonicity in the Sequent Calculus

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Thesis Defense

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My Thesis

Exploiting non-canonicity in the Sequent Calculus

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Exploiting non-canonicity in the Sequent Calculus

- Polarity assignment of literals in focused systems
- Linear logic's exponentials

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Exploiting non-canonicity in the Sequent Calculus

- Polarity assignment of literals in focused systems
- Linear logic's exponentials

- Tabled deduction
- Logical frameworks
- Algorithmic specifications

Agenda

- **Sequent Calculus**
- Focusing
- Tabled Deduction
- Algorithmic Specifications
- Logical Frameworks

Sequent Calculus

Sequent Calculus

$$P_1, \ldots, P_n \vdash Q_1, \ldots, Q_m$$

Sequents

Sequent Calculus

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$$P_1, \ldots, P_n \vdash Q_1, \ldots, Q_m$$

$$P_1 \wedge \cdots \wedge P_n \Rightarrow Q_1 \vee \cdots \vee Q_m$$

Sequent Calculus

Sequents

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$$P_1 \wedge \cdots \wedge P_n \Rightarrow Q_1 \vee \cdots \vee Q_m$$

Inference Rules

$$\frac{}{P \vdash P}\ I$$

$$\frac{\Gamma \vdash P, \Delta \quad \Gamma \vdash Q, \Delta}{\Gamma \vdash P \wedge Q, \Delta}\ \wedge_r$$

$$\frac{\Gamma \vdash P_i, \Delta}{\Gamma \vdash P_1 \vee P_2, \Delta}\ \vee_{ri}$$

Sequent Calculus

Sequents

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$$\frac{\Gamma \vdash P_i, \Delta}{\Gamma \vdash P_1 \vee P_2, \Delta}\ \vee_{ri}$$

Proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\ I}{\vdash A^\perp, A}\ \neg_r}{\vdash A \vee A^\perp, A}\ \vee_{r2}}{\vdash A \vee A^\perp, A \vee A^\perp}\ \vee_{r1}}{\vdash A \vee A^\perp}\ C_r$$

Cut-elimination

$$\frac{\Gamma_1 \vdash P, \Delta_1 \quad \Gamma_2, P \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ Cut$$

Cut-elimination

New formula in the proof. Use of a Lemma.

$$\frac{\Gamma_1 \vdash P, \Delta_1 \qquad \Gamma_2, P \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ Cut$$

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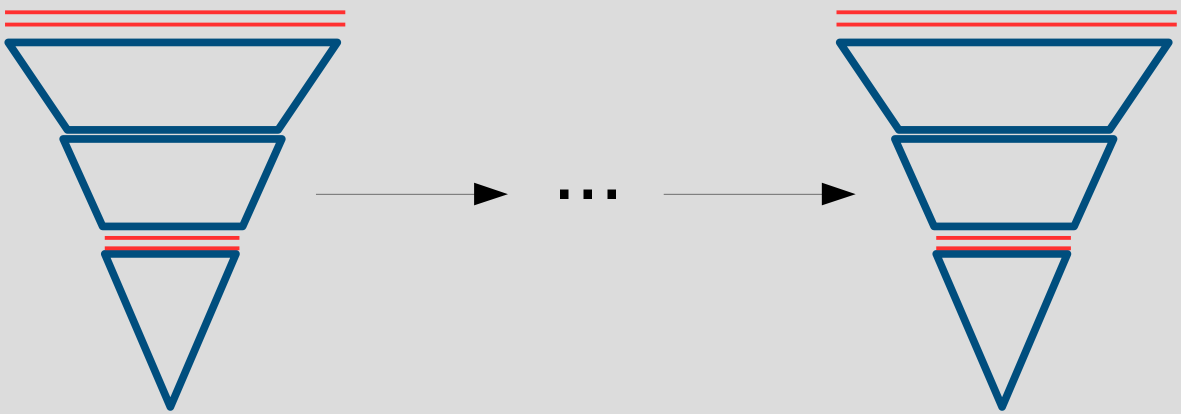
Proof with Cuts

Cut-free Proof



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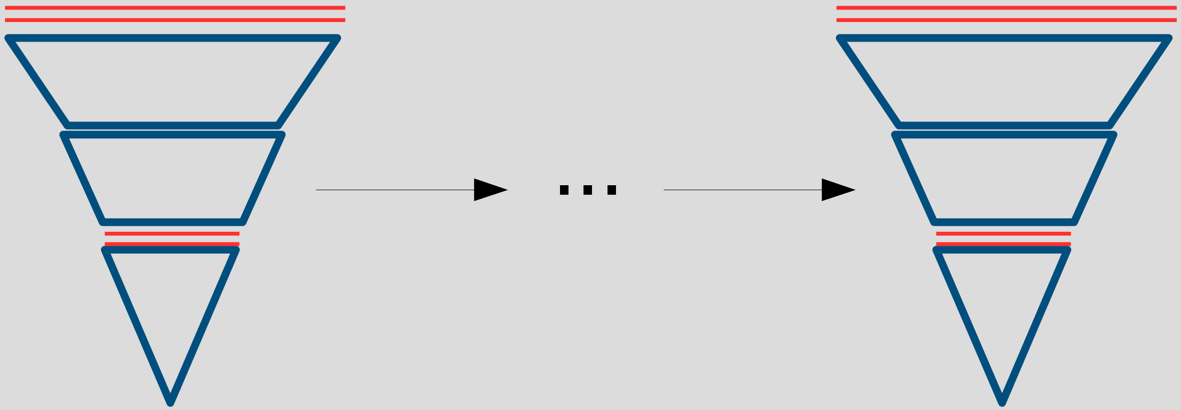
Proof with Cuts

Cut-free Proof



Consequences

1) Consistency

$\vdash P \qquad \vdash P^{\perp}$

2) Subformula Prop.

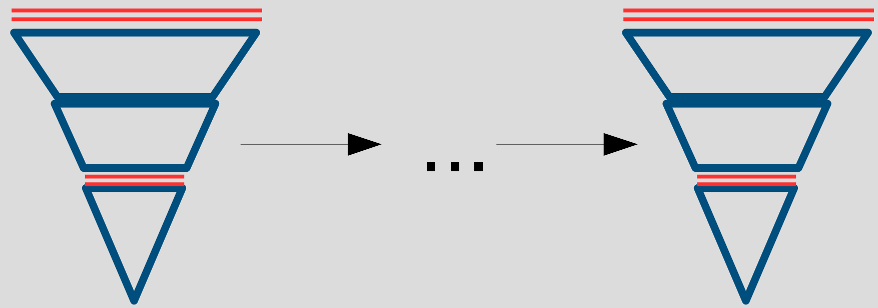
No need for Lemmas

Computational Logic

Computational Logic

Proof normalization

Functional
Programming



Computational Logic

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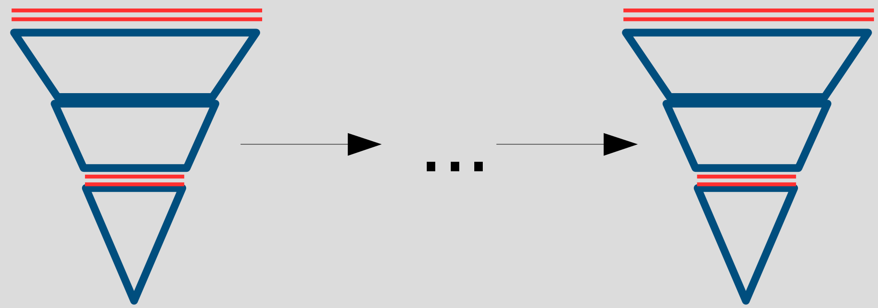
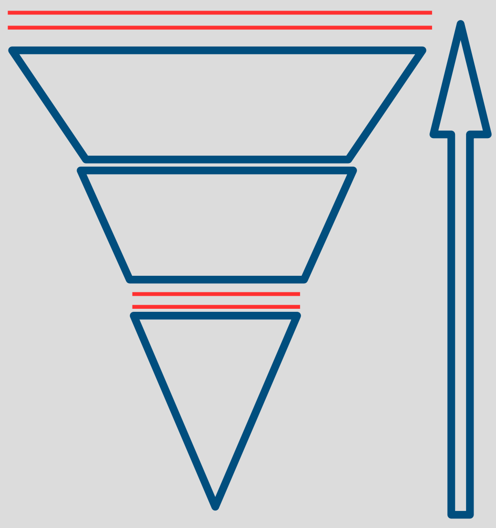


Logic
Programming



Proof search

This thesis

From classical to intuitionistic and linear logics

From classical to intuitionistic and linear logics

Classical Logic

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,I}{\vdash A^{\perp}, A}\,\neg_r}{\vdash A \vee A^{\perp}, A}\,\vee_{r2}}{\vdash A \vee A^{\perp}, A \vee A^{\perp}}\,\vee_{r1}}{\vdash A \vee A^{\perp}}\,C_r$$

Truth

From classical to intuitionistic and linear logics

Classical Logic

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{A \vdash A}\ I}{\vdash A^{\perp}, A}\ \neg_r}{\vdash A \vee A^{\perp}, A}\ \vee_{r2}}{\vdash A \vee A^{\perp}, A \vee A^{\perp}}\ \vee_{r1}}{\vdash A \vee A^{\perp}}\ C_r$$

Truth

Intuitionistic Logic

$\vdash A$ $\quad$ $\vdash A^{\perp}$

$\vdash A \vee A^{\perp}$

Constructive proofs

From classical to intuitionistic and linear logics

Classical Logic

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{A \vdash A}\ I}{\vdash A^{\perp}, A}\ \neg_r}{\vdash A \vee A^{\perp}, A}\ \vee_{r2}}{\vdash A \vee A^{\perp}, A \vee A^{\perp}}\ \vee_{r1}}{\vdash A \vee A^{\perp}}\ C_r$$

Truth

Intuitionistic Logic

$\vdash A$ $\quad$ $\vdash A^{\perp}$

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Constructive proofs

One formula in the right-hand-side

From classical to intuitionistic and linear logics

Classical Logic

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{A \vdash A}\; I}{\vdash A^{\perp}, A}\; \neg_r}{\vdash A \vee A^{\perp}, A}\; \vee_{r2}}{\vdash A \vee A^{\perp}, A \vee A^{\perp}}\; \vee_{r1}}{\vdash A \vee A^{\perp}}\; C_r$$

Truth

Intuitionistic Logic

$\vdash A$ $\qquad$ $\vdash A^{\perp}$

$\vdash A \vee A^{\perp}$

Constructive proofs

Several different sequent calculus systems for these logics

One formula in the right-hand-side

From classical to intuitionistic and linear logics

Linear Logic

Formulas can no longer be used as many times as you want.

From classical to intuitionistic and linear logics

Linear Logic

Formulas can no longer be used as many times as you want.

No canonical form

exponentials $!$ $?$

$$\frac{\vdash \Gamma, ?P, ?P}{\vdash \Gamma, ?P}\, C \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?P}\, W$$

From classical to intuitionistic and linear logics

Linear Logic

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$$\frac{\vdash \Gamma, ?P, ?P}{\vdash \Gamma, ?P}\ C \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?P}\ W$$

$\otimes$	$\perp$
$⅋$	1

multiplicatives

$\oplus$	$\top$
$\&$	0

additives

From classical to intuitionistic and linear logics

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$\otimes$	$\bot$
$⅋$	1

multiplicatives

$\oplus$	$\top$
$\&$	0

additives

$$\frac{\vdash \Gamma, P \quad \vdash \Delta, Q}{\vdash \Gamma, \Delta, P \otimes Q}\ \otimes$$

$$\frac{\vdash \Gamma, P \quad \vdash \Gamma, Q}{\vdash \Gamma, P \,\&\, Q}\ \&$$

From classical to intuitionistic and linear logics

Linear Logic

Formulas can no longer be used as many times as you want.

No canonical form

exponentials $!$ $?$

$$\frac{\vdash \Gamma, ?P, ?P}{\vdash \Gamma, ?P}\ C \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?P}\ W$$

$\otimes$	$\perp$	multiplicatives
$\parr$	1	
$\oplus$	$\top$	additives
$\&$	0	

$$\frac{\vdash \Gamma, P \quad \vdash \Delta, Q}{\vdash \Gamma, \Delta, P \otimes Q}\ \otimes$$

$$\frac{\vdash \Gamma, P \quad \vdash \Gamma, Q}{\vdash \Gamma, P \,\&\, Q}\ \&$$

$A \multimap B$ denotes $B \parr A^\perp$

De Morgan dualities

$\Gamma \vdash \Delta$ $\qquad$ $\vdash \Gamma^\perp, \Delta$

From classical to intuitionistic and linear logics

Linear Logic

The logic behind logic

One can encode intuitionistic logic in linear logic

$$[P \supset Q] \qquad \equiv \qquad ![P] \multimap [Q]$$

From classical to intuitionistic and linear logics

Linear Logic

The logic behind logic

One can encode intuitionistic logic in linear logic

$$[P \supset Q] \qquad \equiv \qquad ![P] \multimap [Q]$$

Logic of resources

One can specify resources

$$\vdash ?(\mathtt{euro}^{\perp} \otimes \mathtt{coffee}), \mathtt{euro}$$

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- **Focusing**
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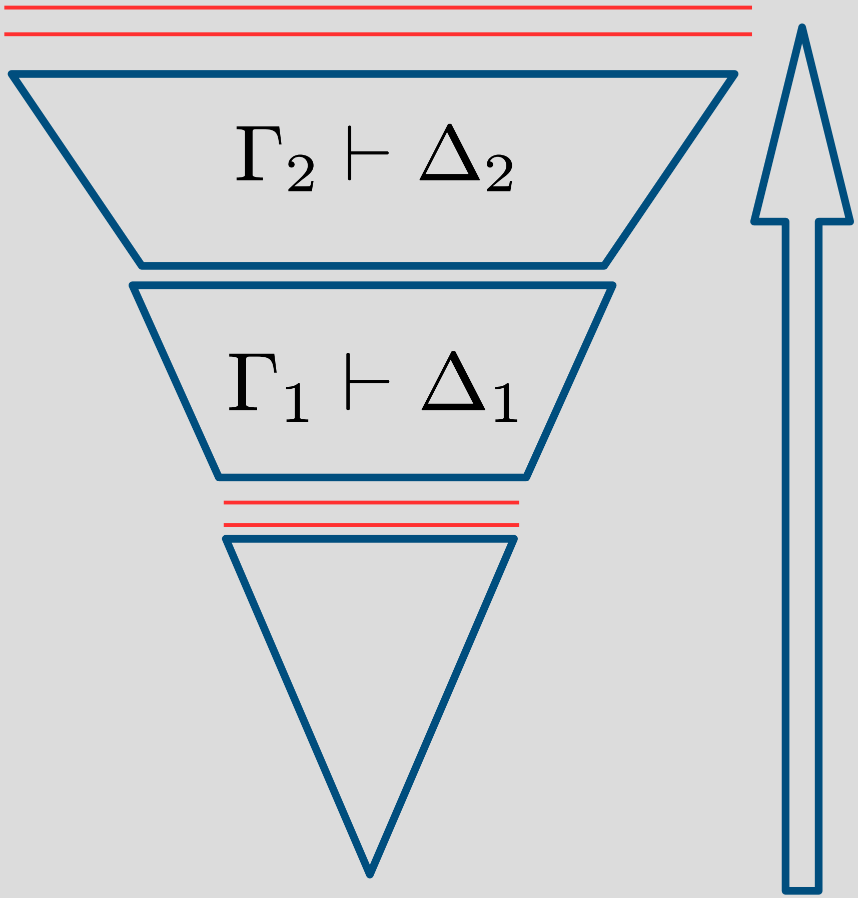
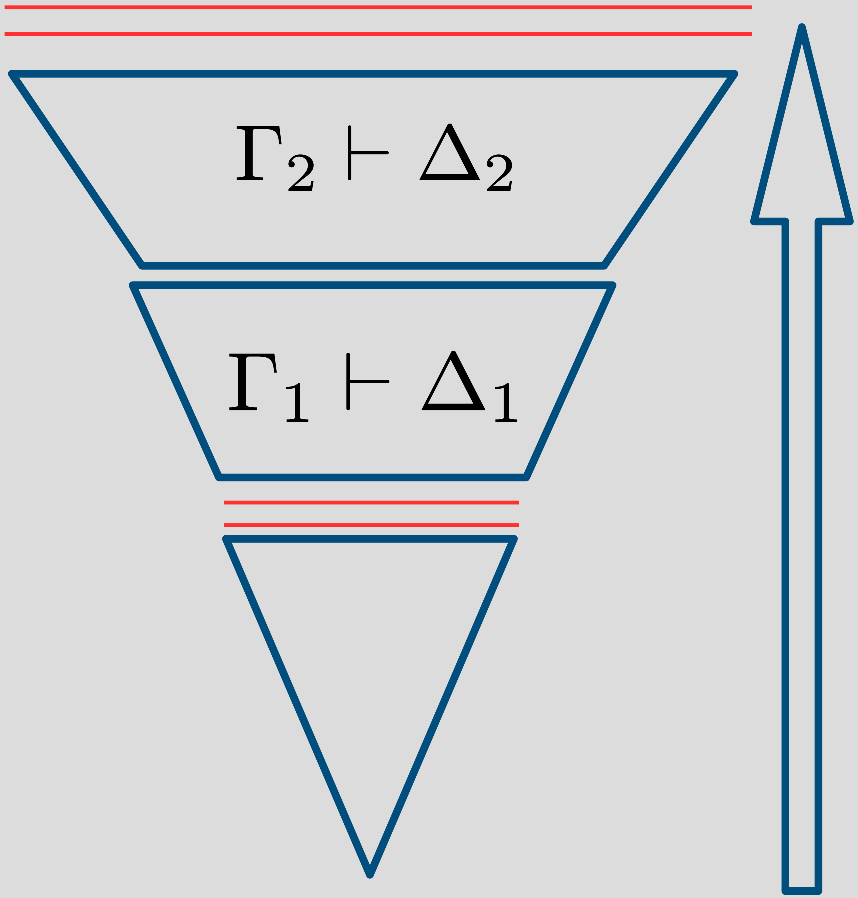
Logic Programming – Search for cut-free proofs



Logic Programming – Search for cut-free proofs



Logic program

$\forall x(\mathsf{path}\; x\; x)$

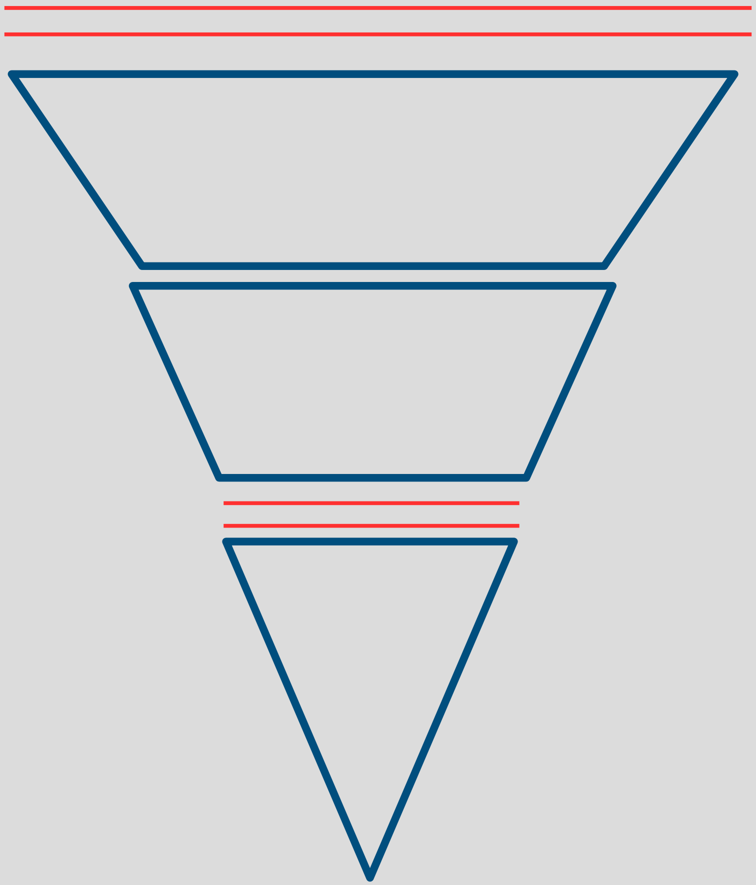
$\forall x \forall y \forall z(\mathsf{arr}\; x\; z \wedge \mathsf{path}\; z\; y \supset \mathsf{path}\; x\; y)$

Query

$\mathsf{path}\; a_3\; a_4$

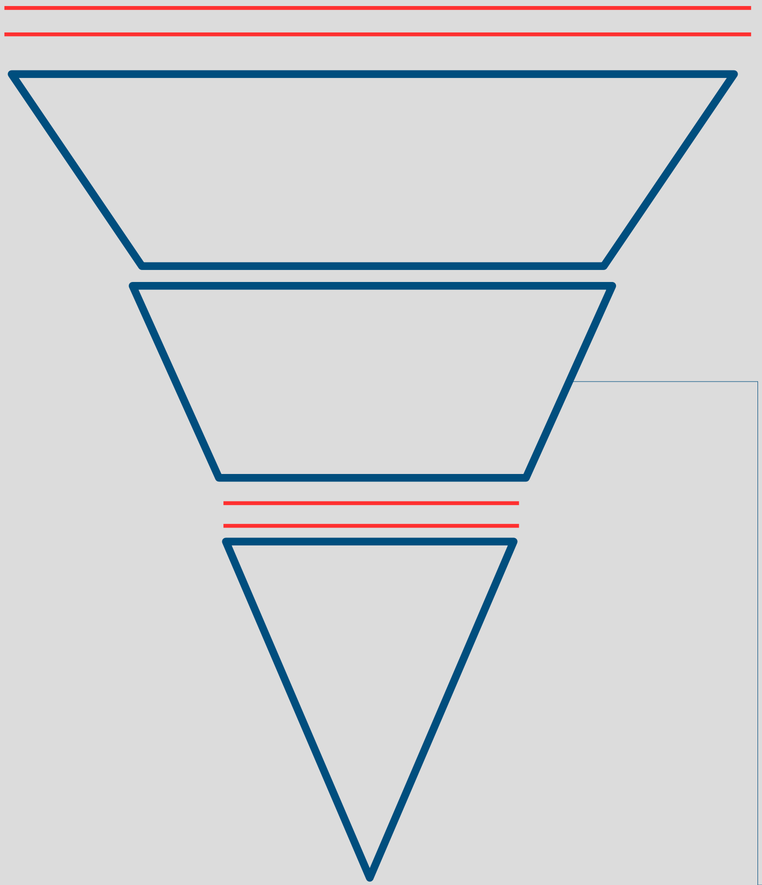
Focusing

Focusing



Normal form
proofs for
proof search

Focusing

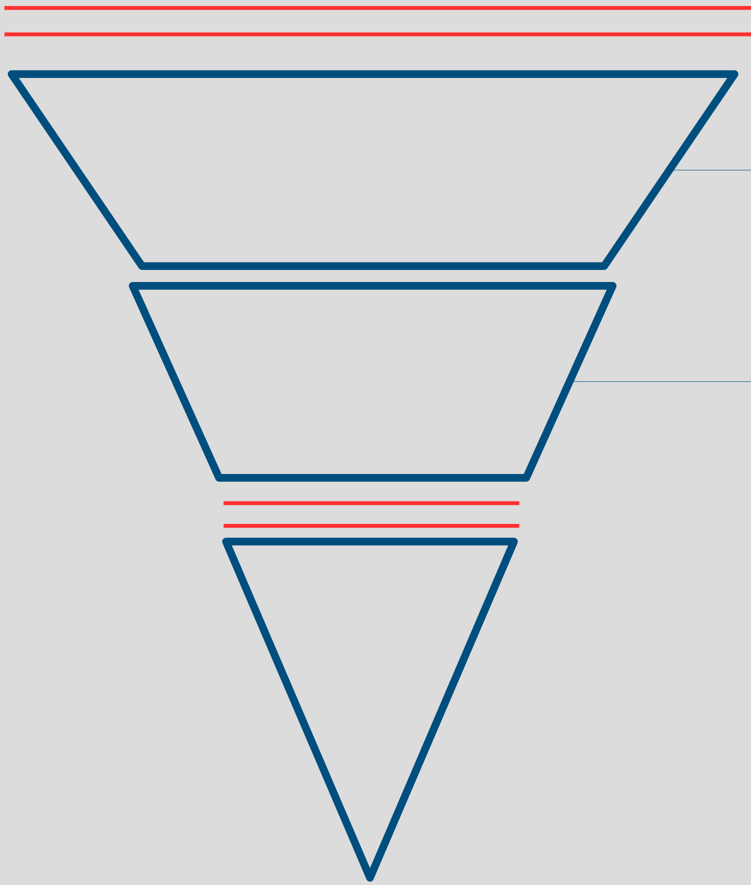


Normal form
proofs for
proof search

Negative Phase - All invertible rules are applied eagerly

$$\frac{\vdash \Theta : \Gamma \Uparrow L, F, G}{\vdash \Theta : \Gamma \Uparrow L, F \parr G} \; [\parr]$$

Focusing



Normal form proofs for proof search

Positive Phase – One formula is focused on

$$\frac{\vdash \Theta : \Gamma \Downarrow P}{\vdash \Theta : \Gamma, P \Uparrow} \; [D_1]$$

Focusing persists

$$\frac{\vdash \Theta : \Gamma \Downarrow F \quad \vdash \Theta : \Gamma' \Downarrow G}{\vdash \Theta : \Gamma, \Gamma' \Downarrow F \otimes G} \; [\otimes]$$

Negative Phase - All invertible rules are applied eagerly

$$\frac{\vdash \Theta : \Gamma \Uparrow L, F, G}{\vdash \Theta : \Gamma \Uparrow L, F \parr G} \; [\parr]$$

Focusing Basics

$A \mathbin{\&} B, A \parr B, \bot, \top, ?B, \forall x\, B$ Negative Formulas

Focusing Basics

$A \mathbin{\&} B, A \parr B, \bot, \top, ?B, \forall x\, B$ Negative Formulas

$$\frac{\vdash \Theta : \Gamma \Uparrow L}{\vdash \Theta : \Gamma \Uparrow L, \bot}\ [\bot] \qquad \frac{\vdash \Theta : \Gamma \Uparrow L, F, G}{\vdash \Theta : \Gamma \Uparrow L, F \parr G}\ [\parr] \qquad \frac{\vdash \Theta, F : \Gamma \Uparrow L}{\vdash \Theta : \Gamma \Uparrow L, ?F}\ [?]$$

$$\frac{}{\vdash \Theta : \Gamma \Uparrow L, \top}\ [\top] \qquad \frac{\vdash \Theta : \Gamma \Uparrow L, F \quad \vdash \Theta : \Gamma \Uparrow L, G}{\vdash \Theta : \Gamma \Uparrow L, F \mathbin{\&} G}\ [\&] \qquad \frac{\vdash \Theta : \Gamma \Uparrow L, F[c/x]}{\vdash \Theta : \Gamma \Uparrow L, \forall x\, F}\ [\forall]$$

All negative rules are invertible

Focusing Basics

$A \otimes B, A \oplus B, 1, !\, B, \exists x\, B$ Positive Formulas

Focusing Basics

$$A \otimes B, A \oplus B, 1, !\, B, \exists x\, B$$

Positive Formulas

$$\frac{}{\vdash \Theta :\Downarrow 1}\ [1] \qquad \frac{\vdash \Theta : \Gamma \Downarrow F \quad \vdash \Theta : \Gamma' \Downarrow G}{\vdash \Theta : \Gamma, \Gamma' \Downarrow F \otimes G}\ [\otimes] \qquad \frac{\vdash \Theta :\Uparrow F}{\vdash \Theta :\Downarrow !\, F}\ [!]$$

$$\frac{\vdash \Theta : \Gamma \Downarrow F}{\vdash \Theta : \Gamma \Downarrow F \oplus G}\ [\oplus_l] \qquad \frac{\vdash \Theta : \Gamma \Downarrow G}{\vdash \Theta : \Gamma \Downarrow F \oplus G}\ [\oplus_r] \qquad \frac{\vdash \Theta, F : \Gamma \Downarrow F[t/x]}{\vdash \Theta : \Gamma \Downarrow \exists x\, F}\ [\exists]$$

Positive rules are not necessarily invertible.

Focusing Basics

Structural Rules

$$\frac{\vdash \Theta : \Gamma \Uparrow N}{\vdash \Theta : \Gamma \Downarrow N}\ [R\Downarrow] \qquad \frac{\vdash \Theta : \Gamma, S \Uparrow L}{\vdash \Theta : \Gamma \Uparrow L, S}\ [R\Uparrow]$$

$$\frac{\vdash \Theta : \Gamma \Downarrow P}{\vdash \Theta : \Gamma, P \Uparrow}\ [D_1] \qquad \frac{\vdash \Theta, P : \Gamma \Downarrow P}{\vdash \Theta, P : \Gamma \Uparrow}\ [D_2]$$

Here, *N* is a negative formula, *P* is not a negative literal, and *S* is a positive formula or a literal.

Focusing Basics

Synthetic Connectives

$$\dfrac{\vdash \Theta : \Gamma \Uparrow A_1}{\vdash \Theta : \Gamma \Downarrow A_1 \oplus (A_2 \otimes A_3)} \qquad \dfrac{\vdash \Theta : \Gamma_1 \Uparrow A_2 \quad \vdash \Theta : \Gamma_2 \Uparrow A_3}{\vdash \Theta : \Gamma_1, \Gamma_2 \Downarrow A_1 \oplus (A_2 \otimes A_3)}$$

Focusing Basics

Synthetic Connectives

$$\frac{\vdash \Theta : \Gamma \Uparrow A_1}{\vdash \Theta : \Gamma \Downarrow A_1 \oplus (A_2 \otimes A_3)} \qquad \frac{\vdash \Theta : \Gamma_1 \Uparrow A_2 \qquad \vdash \Theta : \Gamma_2 \Uparrow A_3}{\vdash \Theta : \Gamma_1, \Gamma_2 \Downarrow A_1 \oplus (A_2 \otimes A_3)}$$

We can construct “macro-rules” that introduce the synthetic connectives of formulas.

Focusing Basics

Literals are arbitrarily classified as positive or negative

Focusing Basics

Literals are arbitrarily classified as positive or negative

$$\frac{}{\vdash \Theta : A_p^{\perp} \Downarrow A_p} \; [I_1] \qquad \frac{}{\vdash \Theta, A_p^{\perp} :\Downarrow A_p} \; [I_2]$$

Focusing Basics

Literals are arbitrarily classified as positive or negative

$$\frac{}{\vdash \Theta : A_p^{\perp} \Downarrow A_p}\ [I_1] \qquad \frac{}{\vdash \Theta, A_p^{\perp} :\Downarrow A_p}\ [I_2]$$

The Focusing Theorem states that a formula is provable in the focused system iff it is provable in linear logic. Does not matter how we assign the polarity of literals.

Playing with polarities

Fibonacci Program	To prove
$\mathrm{fib}(0,0) \wedge \mathrm{fib}(1,1) \wedge$ $\forall n, f, f' [\mathrm{fib}(n, f) \supset \mathrm{fib}(n+1, f') \supset \mathrm{fib}(n+2, f+f')].$	$\Gamma \longrightarrow \mathrm{fib}(n, N).$

Playing with polarities

Fibonacci Program

$$\text{fib}(0,0) \wedge \text{fib}(1,1) \wedge$$
$$\forall n, f, f' [\text{fib}(n, f) \supset \text{fib}(n+1, f') \supset \text{fib}(n+2, f+f')].$$

To prove

$$\Gamma \longrightarrow \text{fib}(n, N).$$

fib atoms as negative

there is a unique focused proof of size exponential in ***n*** (goal-directed, backchaining)

Playing with polarities

Fibonacci Program

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fib atoms as negative

there is a unique focused proof of size exponential in ***n*** (goal-directed, backchaining)

fib atoms as positive

there are infinitely many proofs and the smallest one is of linear size in ***n*** (program-directed, forward-chaining).

Playing with polarities

Fibonacci Program

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$$\forall n, f, f'[\text{fib}(n,f) \supset \text{fib}(n+1,f') \supset \text{fib}(n+2,f+f')].$$

To prove

$$\Gamma \longrightarrow \text{fib}(n,N).$$

fib atoms as negative

there is a unique focused proof of size exponential in ***n*** (goal-directed, backchaining)

fib atoms as positive

there are infinitely many proofs and the smallest one is of linear size in ***n*** (program-directed, forward-chaining).

While choices in the polarization of atoms **do not affect provability**, it can have important consequences on the shape of proofs.

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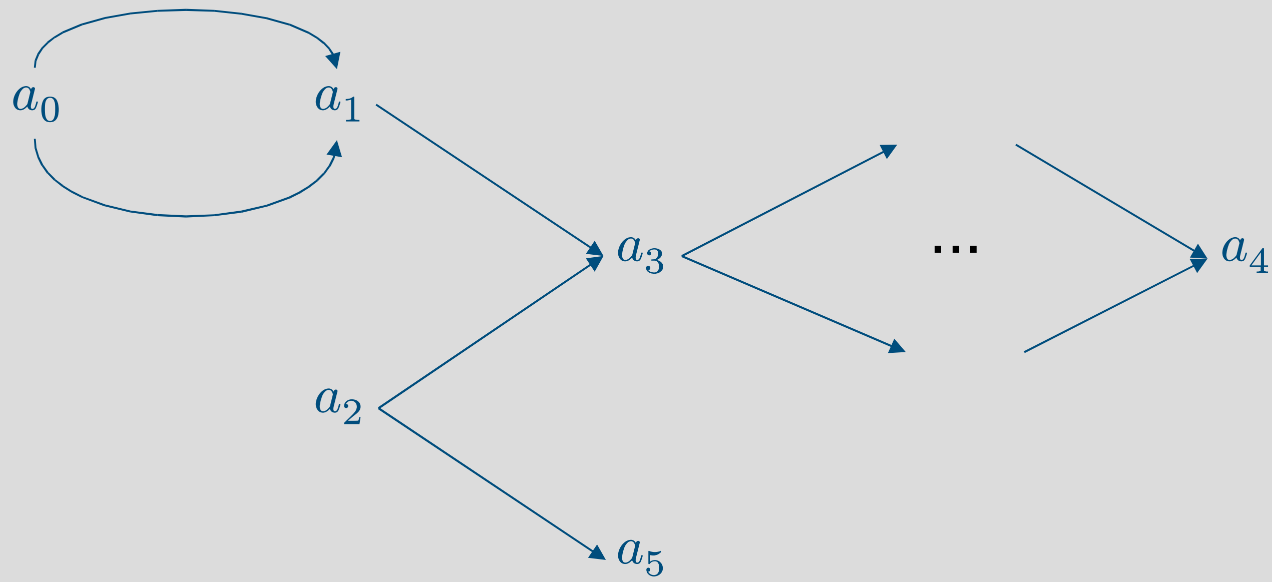
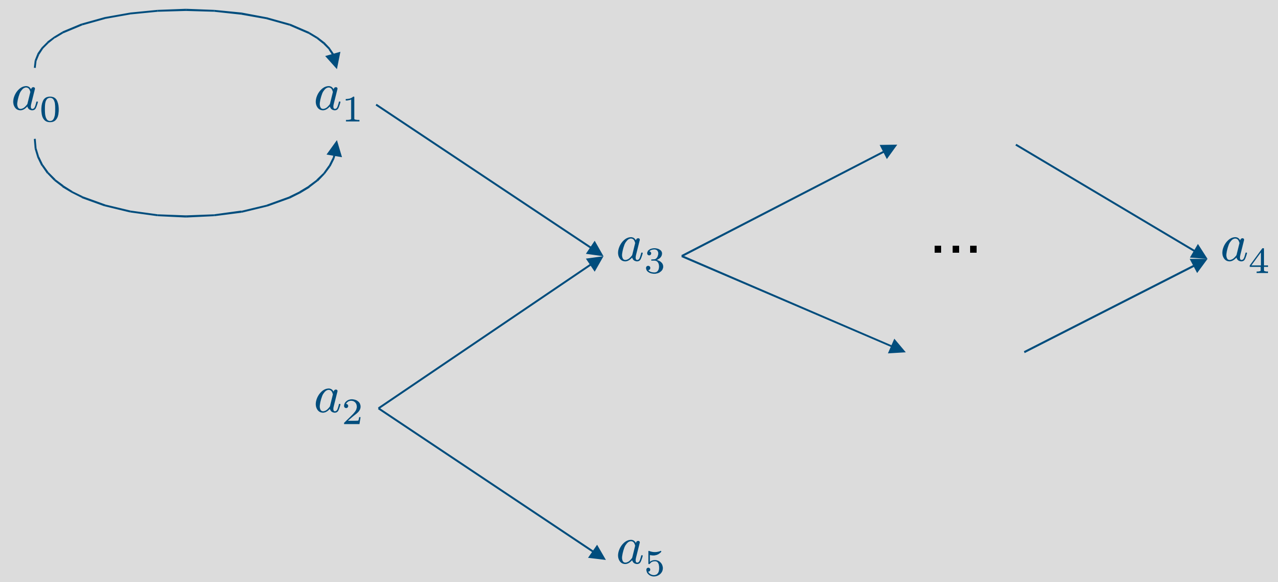
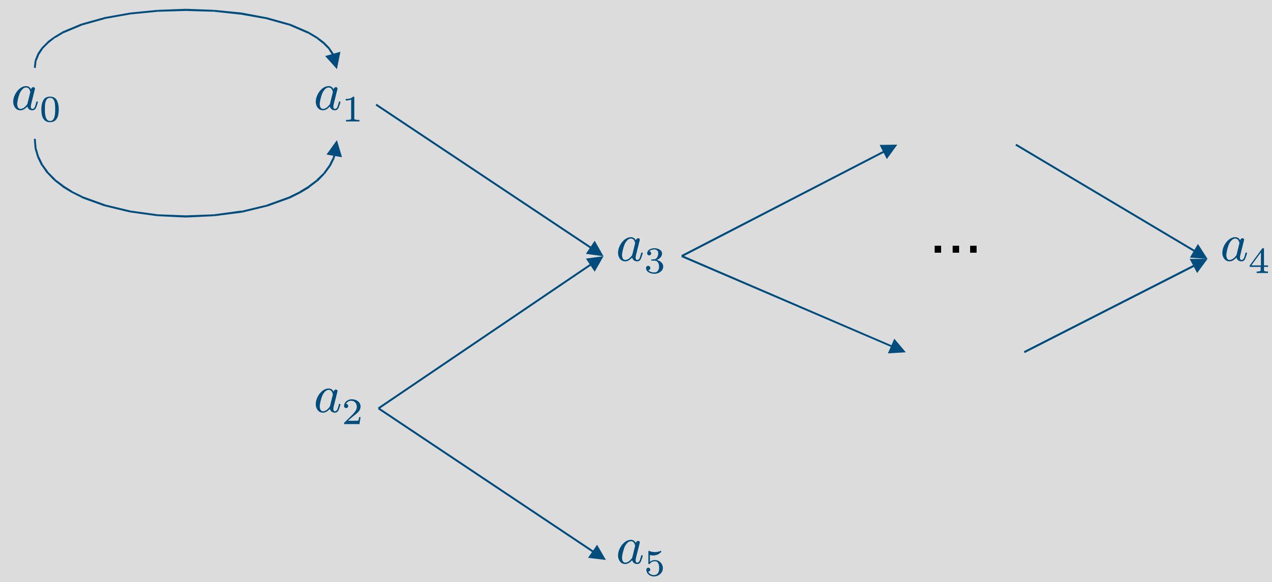

a_0
a_1
a_2
a_3
...
a_4
a_5



$\forall x(\mathbf{path}\, x\, x)$
$\forall x \forall y \forall z(arr\ x\ z \wedge \mathbf{path}\ z\ y \supset \mathbf{path}\ x\ y)$

$$\mathbf{path}\ a_1\ a_4 \wedge \mathbf{path}\ a_2\ a_4$$



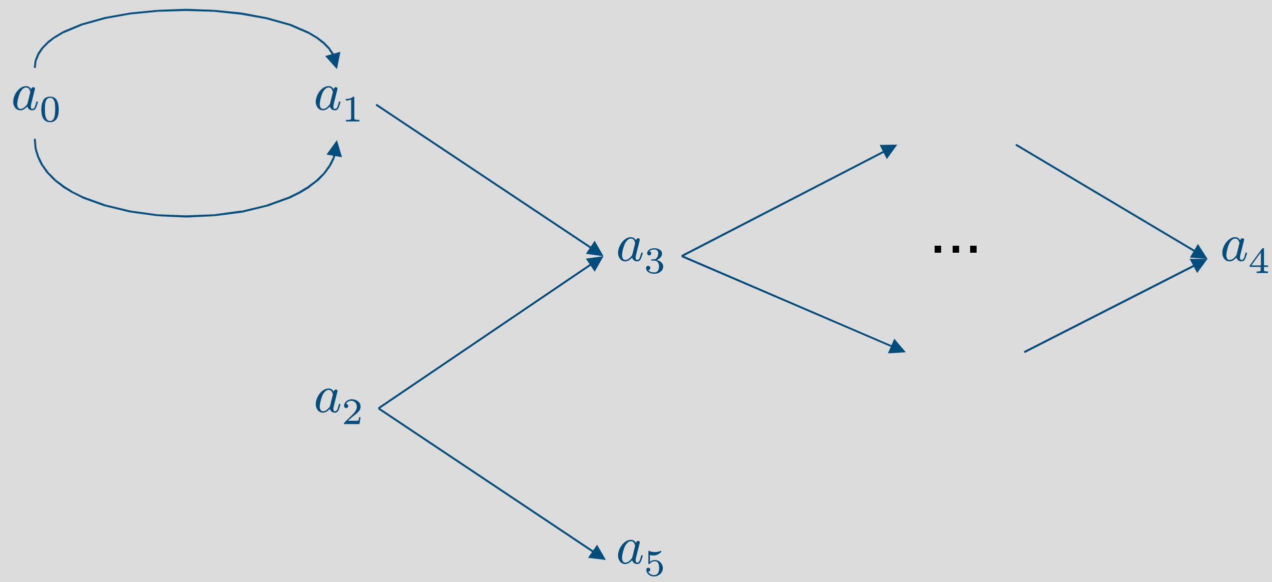
$\forall x(\mathbf{path}\, x\, x)$
$\forall x \forall y \forall z(\mathrm{arr}\ x\ z \wedge \mathbf{path}\ z\ y \supset \mathbf{path}\ x\ y)$

$$\mathbf{path}\ a_1\ a_4 \wedge \mathbf{path}\ a_2\ a_4$$

Common Subgoal

$$\mathbf{path}\ a_3\ a_4$$

- In Prolog, this common subgoal is computed twice.



$\forall x(\mathbf{path}\, x\, x)$
$\forall x \forall y \forall z(arr\ x\ z \wedge \mathbf{path}\ z\ y \supset \mathbf{path}\ x\ y)$

$\mathbf{path}\ a_2\ a_5$

Two paths:

- One is an expensive failure and the other an easy success

How to avoid this **declaratively**:

How to avoid this declaratively:

- Introduce the common subgoal with a cut

$$\frac{\Gamma \vdash A \qquad A, \Gamma \vdash A \wedge G}{\Gamma \vdash A \wedge G}$$

How to avoid this declaratively:

- Introduce the common subgoal with a cut

$$\frac{\Gamma \vdash A \qquad A, \Gamma \vdash A \wedge G}{\Gamma \vdash A \wedge G}$$

Another example (without cuts)

- Change to an equivalent goal:

$$A \wedge G \equiv A \wedge (A \supset G)$$

How to avoid this declaratively:

- Introduce the common subgoal with a cut

$$\frac{\Gamma \vdash A \qquad A, \Gamma \vdash A \wedge G}{\Gamma \vdash A \wedge G}$$

Another example (without cuts)

- Change to an equivalent goal:

$$A \wedge G \equiv A \wedge (A \supset G)$$

But we are only increasing non-determinism:

- There are now more proofs for the goal;
- How to give a purely **proof theoretic solution** where common subgoals aren't re-proven see Chapter 3 of my thesis.

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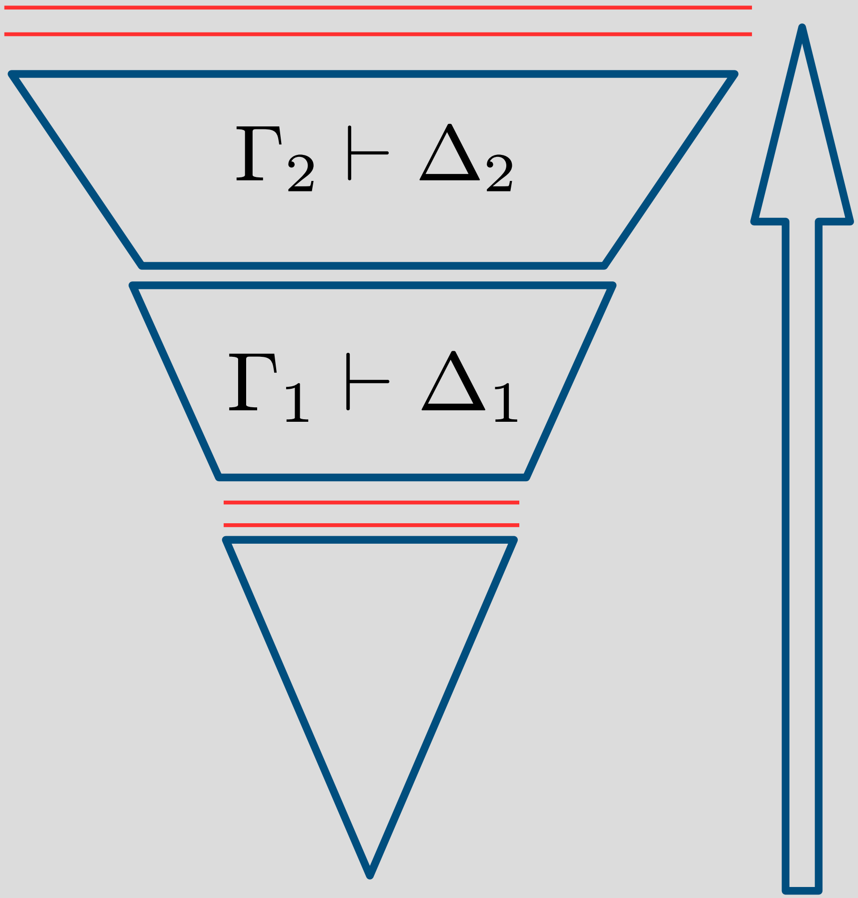
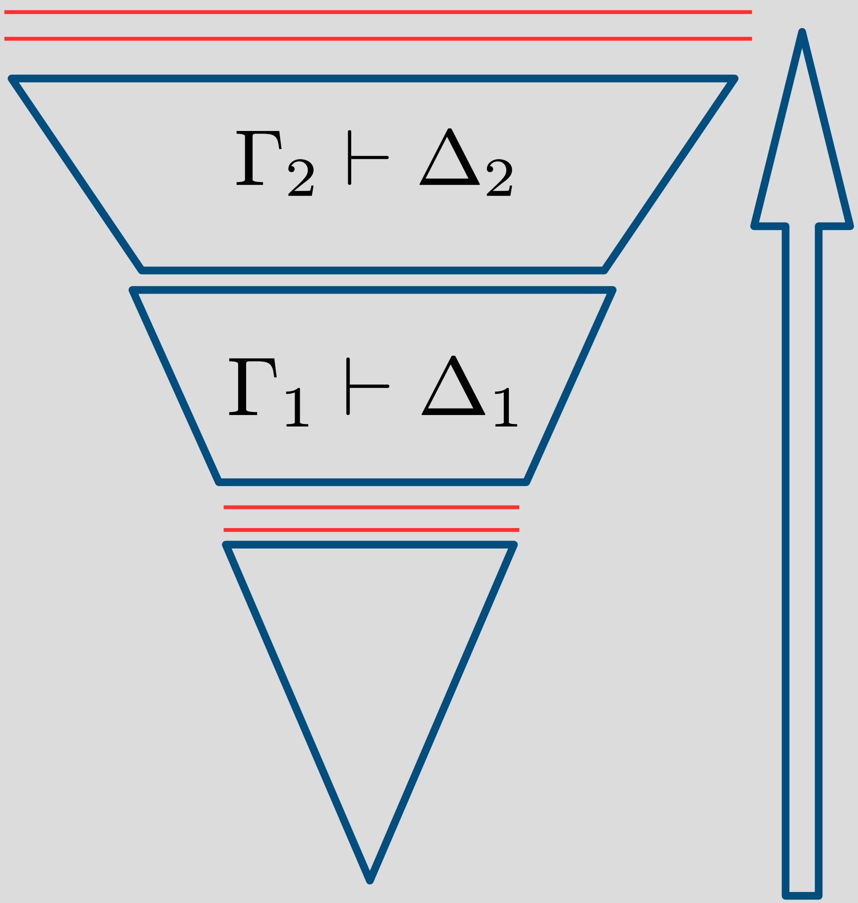
Logic Programming - Dynamics of Proof Search



Logic Programming - Dynamics of Proof Search

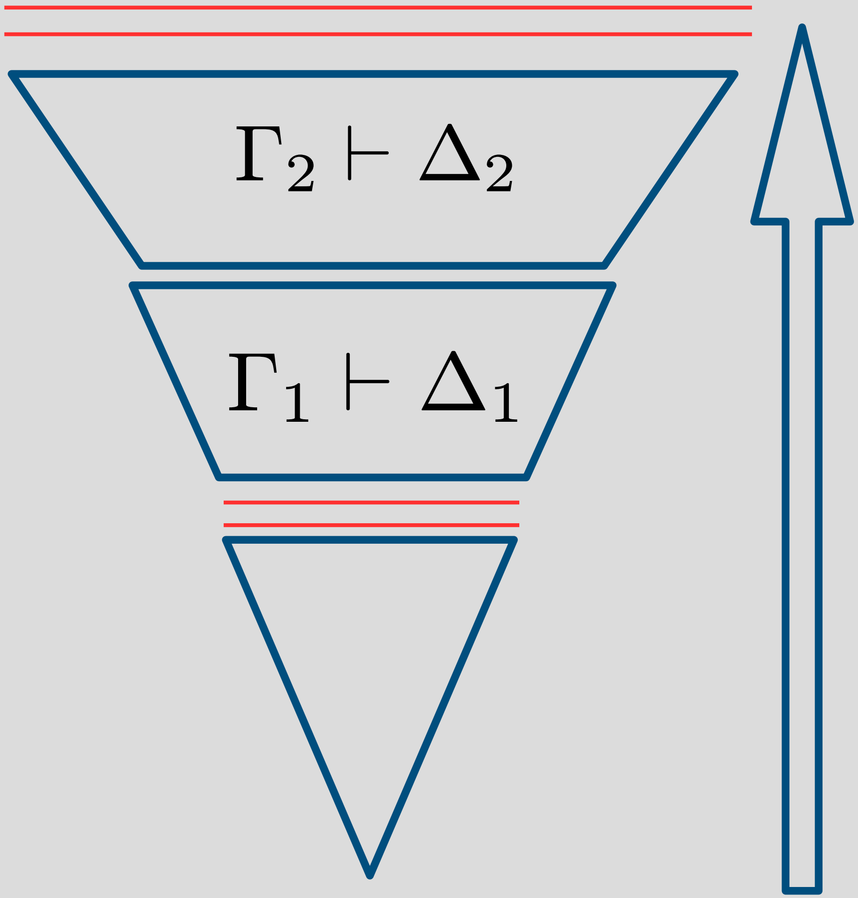


Horn Theory (hHf)

$\Gamma_1 = \Gamma_2 \quad (\Gamma_1 \subseteq \Gamma_2)$

Δ_1, Δ_2 are atomic

Logic Programming - Dynamics of Proof Search



Horn Theory (hHf)

$$\Gamma_1 = \Gamma_2 \quad (\Gamma_1 \subseteq \Gamma_2\ )$$

Δ_1, Δ_2 are atomic

Computation in level of terms:

$$min(X :: nil, X).$$

$$min(X :: L, Y) : -X > Y, min(L, Y)$$

$$min(X :: L, X) : -X <= Y, min(L, Y)$$

Logic Programming - Dynamics of Proof Search

Linear Logic

$\Gamma_1, \Gamma_2, \Delta_1, \Delta_2$

are multisets.

Logic Programming - Dynamics of Proof Search

Linear Logic

$\Gamma_1, \Gamma_2, \Delta_1, \Delta_2$

are multisets.

A representation of a graph:

$$N = \{\mathsf{node}\ x \mid x \in \mathcal{N}\}$$
$$A = \{\mathsf{adj}\ x\ y \mid \langle x, y\rangle \in \mathcal{A}\}$$

$$\vdash N, A$$

Logic Programming - Dynamics of Proof Search

Linear Logic

$\Gamma_1, \Gamma_2, \Delta_1, \Delta_2$

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$$\vdash N, A$$

How to check that only the set *N* is empty?

Logic Programming - Dynamics of Proof Search

Linear Logic

$\Gamma_1, \Gamma_2, \Delta_1, \Delta_2$

are multisets.

A representation of a graph:

$$N = \{\texttt{node}\; x \mid x \in \mathcal{N}\}$$
$$A = \{\texttt{adj}\; x\; y \mid \langle x, y\rangle \in \mathcal{A}\}$$

$$\vdash N, A$$

How to check that only the set *N* is empty?

Not so easy!

We are able to check if the **whole** linear context is empty:

$$\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !\, A}\ [!]$$

Logic Programming - Dynamics of Proof Search

Linear Logic

$\Gamma_1, \Gamma_2, \Delta_1, \Delta_2$

are multisets.

A representation of a graph:

$$N = \{\textsf{node}\ x \mid x \in \mathcal{N}\}$$
$$A = \{\texttt{adj}\ x\ y \mid \langle x, y\rangle \in \mathcal{A}\}$$

$$\vdash N, A$$

How to check that only the set *N* is empty?

Not so easy!

We are able to check if the **whole** linear context is empty:

$$\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !\,A}\ [!]$$

We need local contexts.
We need subexponentials.

Subexponentials

$?^{b}, !^{b}$ $?^{r}, !^{r}$

Subexponentials

$?^{\mathsf{b}}, !^{\mathsf{b}}$ $\qquad$ $?^{\mathsf{r}}, !^{\mathsf{r}}$

not provable

$!^{\mathsf{b}}F \equiv !^{\mathsf{r}}F \qquad ?^{\mathsf{b}}F \equiv ?^{\mathsf{r}}F$

Subexponentials

$?^{\mathsf{b}}, !^{\mathsf{b}}$ $\quad$ $?^{\mathsf{r}}, !^{\mathsf{r}}$

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$!^{\mathsf{b}}F \equiv !^{\mathsf{r}}F \quad ?^{\mathsf{b}}F \equiv ?^{\mathsf{r}}F$

Subexp Signature $\langle I, \preceq, \mathcal{W}, \mathcal{C} \rangle$

$\mathcal{W}$ and $\mathcal{C}$ are up. closed under $\preceq$

Subexponentials

$?^{\mathsf{b}}, !^{\mathsf{b}} \qquad ?^{\mathsf{r}}, !^{\mathsf{r}}$

not provable

$!^{\mathsf{b}} F \equiv !^{\mathsf{r}} F \qquad ?^{\mathsf{b}} F \equiv ?^{\mathsf{r}} F$

Subexp Signature $\langle I, \preceq, \mathcal{W}, \mathcal{C} \rangle$

$\mathcal{W}$ and $\mathcal{C}$ are up. closed under $\preceq$

$$\mathsf{y} \in \mathcal{C} \text{ and } \mathsf{z} \in \mathcal{W}$$

$$\frac{\vdash C, \Delta}{\vdash ?^{\mathsf{x}} C, \Delta}\ D \qquad \frac{\vdash ?^{\mathsf{y}} C, ?^{\mathsf{y}} C, \Delta}{\vdash ?^{\mathsf{y}} C, \Delta}\ C \qquad \frac{\vdash, \Delta}{\vdash ?^{\mathsf{z}} C, \Delta}\ W$$

Subexponentials

$?^{\mathsf{b}}, !^{\mathsf{b}}$ $\quad ?^{\mathsf{r}}, !^{\mathsf{r}}$

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$!^{\mathsf{b}}F \equiv !^{\mathsf{r}}F \quad ?^{\mathsf{b}}F \equiv ?^{\mathsf{r}}F$

Subexp Signature $\langle I, \preceq, \mathcal{W}, \mathcal{C} \rangle$

$\mathcal{W}$ and $\mathcal{C}$ are up. closed under $\preceq$

$\mathsf{y} \in \mathcal{C}$ and $\mathsf{z} \in \mathcal{W}$

$$\frac{\vdash C, \Delta}{\vdash ?^{\mathsf{x}}C, \Delta}\ D \quad \frac{\vdash ?^{\mathsf{y}}C, ?^{\mathsf{y}}C, \Delta}{\vdash ?^{\mathsf{y}}C, \Delta}\ C \quad \frac{\vdash, \Delta}{\vdash ?^{\mathsf{z}}C, \Delta}\ W$$

$$\frac{\vdash ?^{x_1}C_1, \ldots, ?^{x_n}C_n, C}{\vdash ?^{x_1}C_1, \ldots, ?^{x_n}C_n, !^a C}\ !^a$$

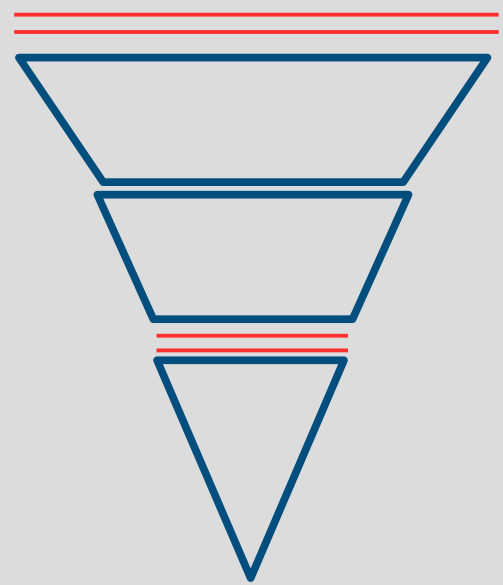
we can now check if a subset is empty

$$a \preceq x_i \text{ for all } i = 1, \ldots, n$$

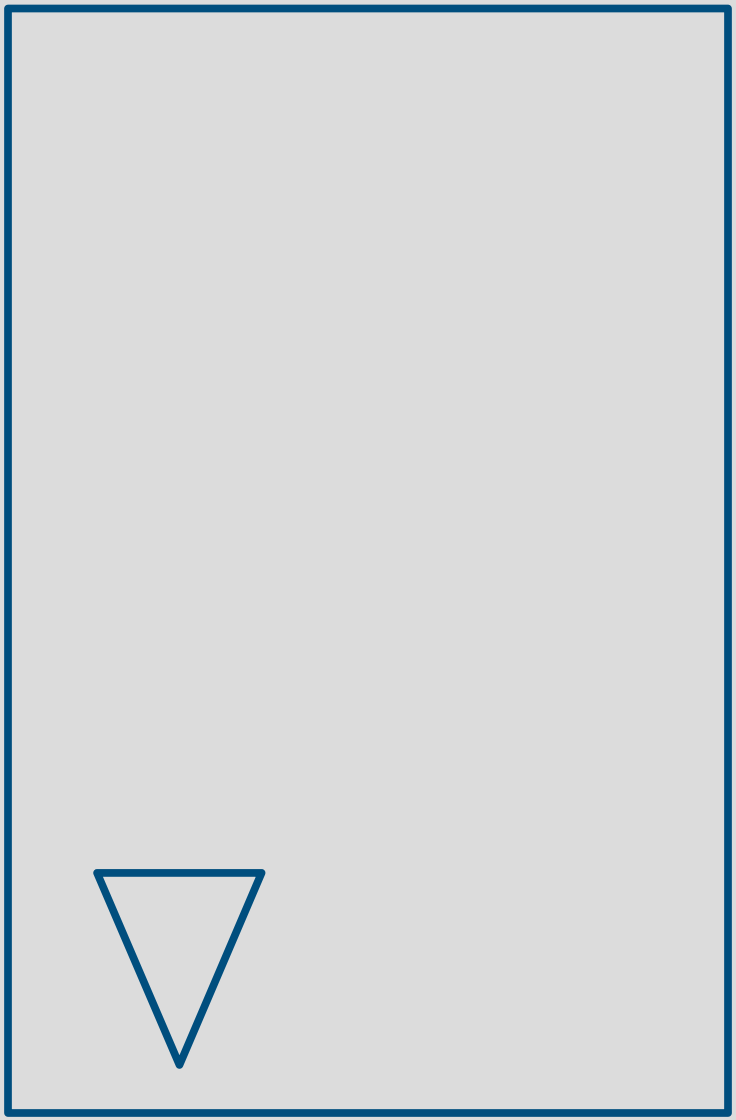
Focused proof search yields algorithmic specification

Focused proof search yields algorithmic specification

Final Proof



Focused proof search yields algorithmic specification



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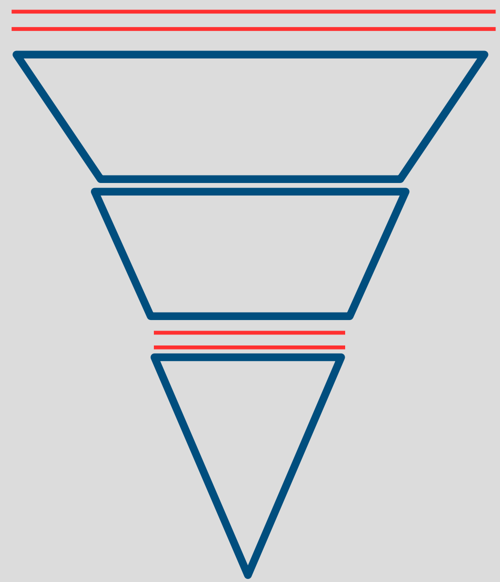

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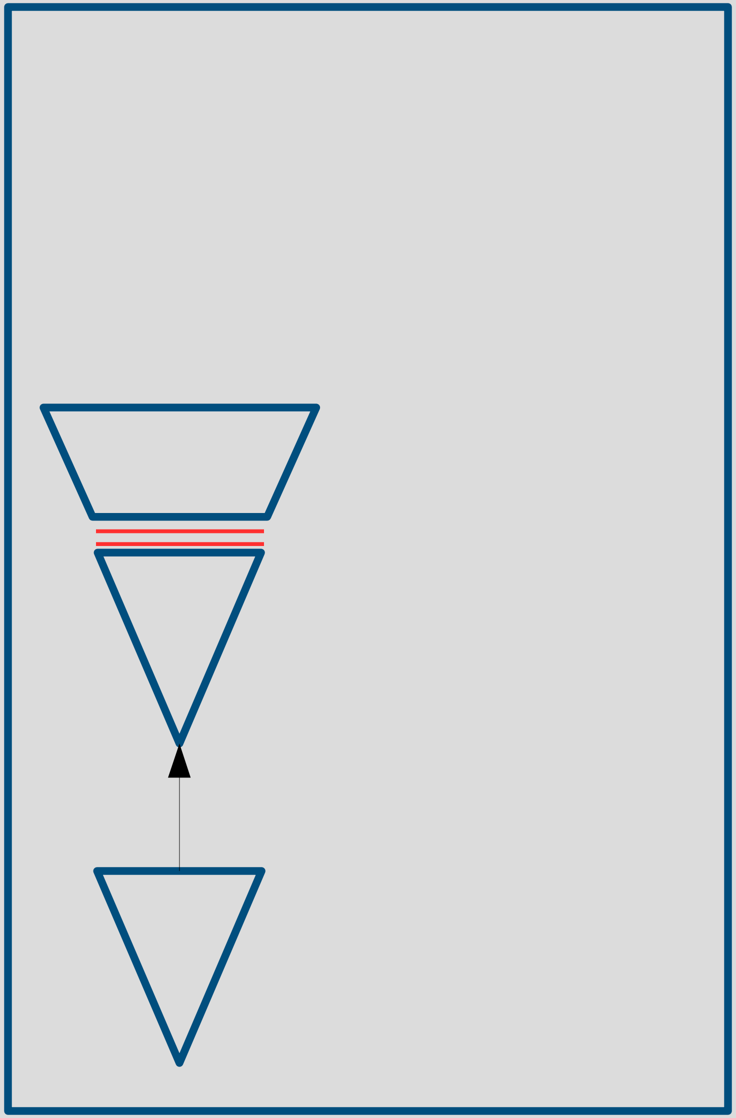
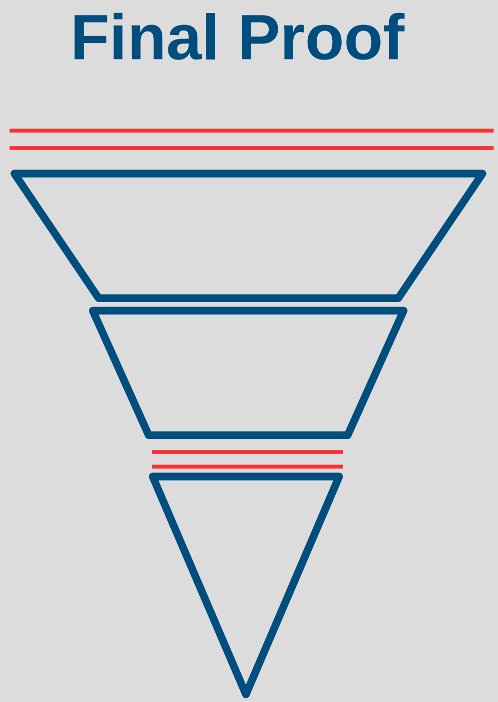
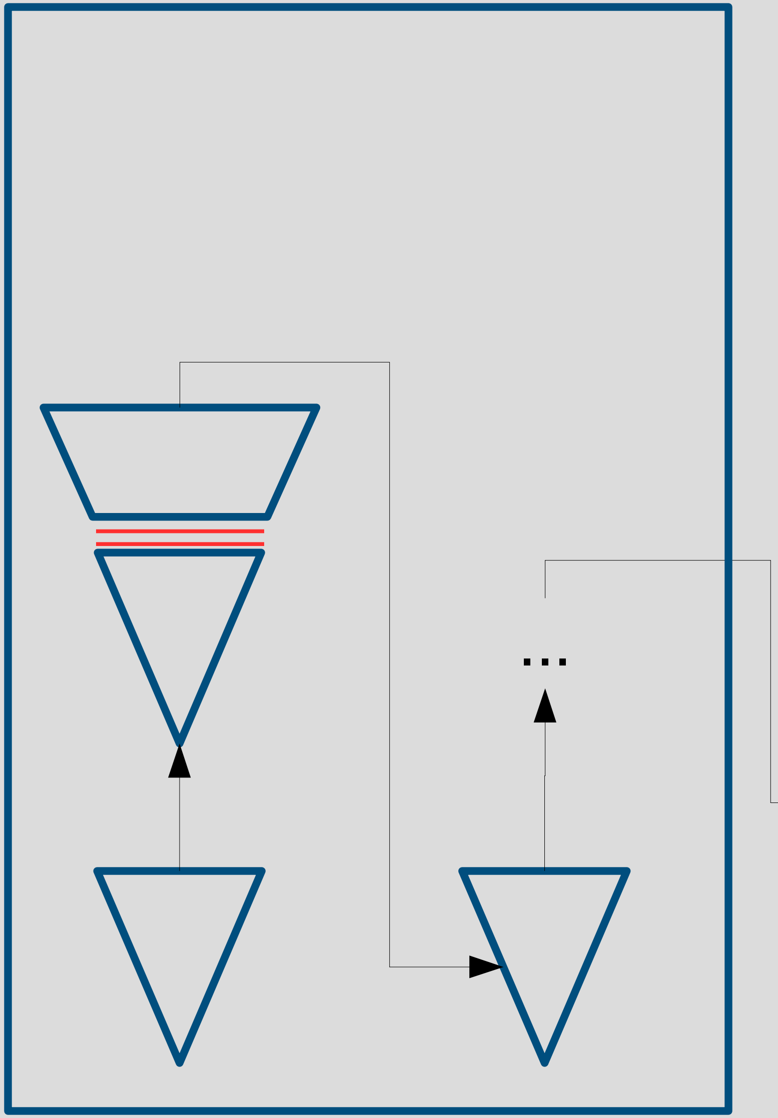
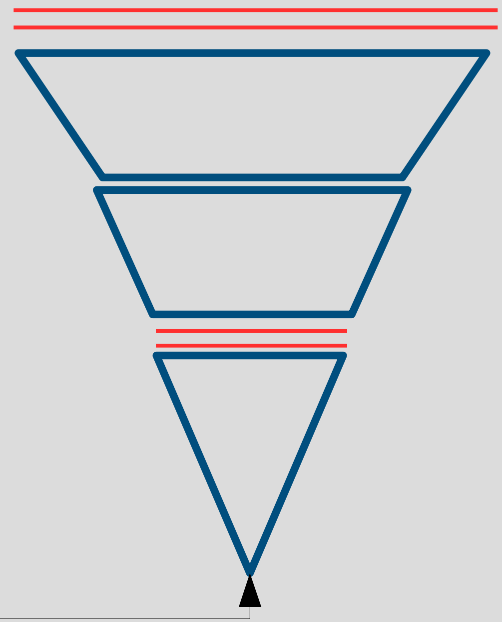




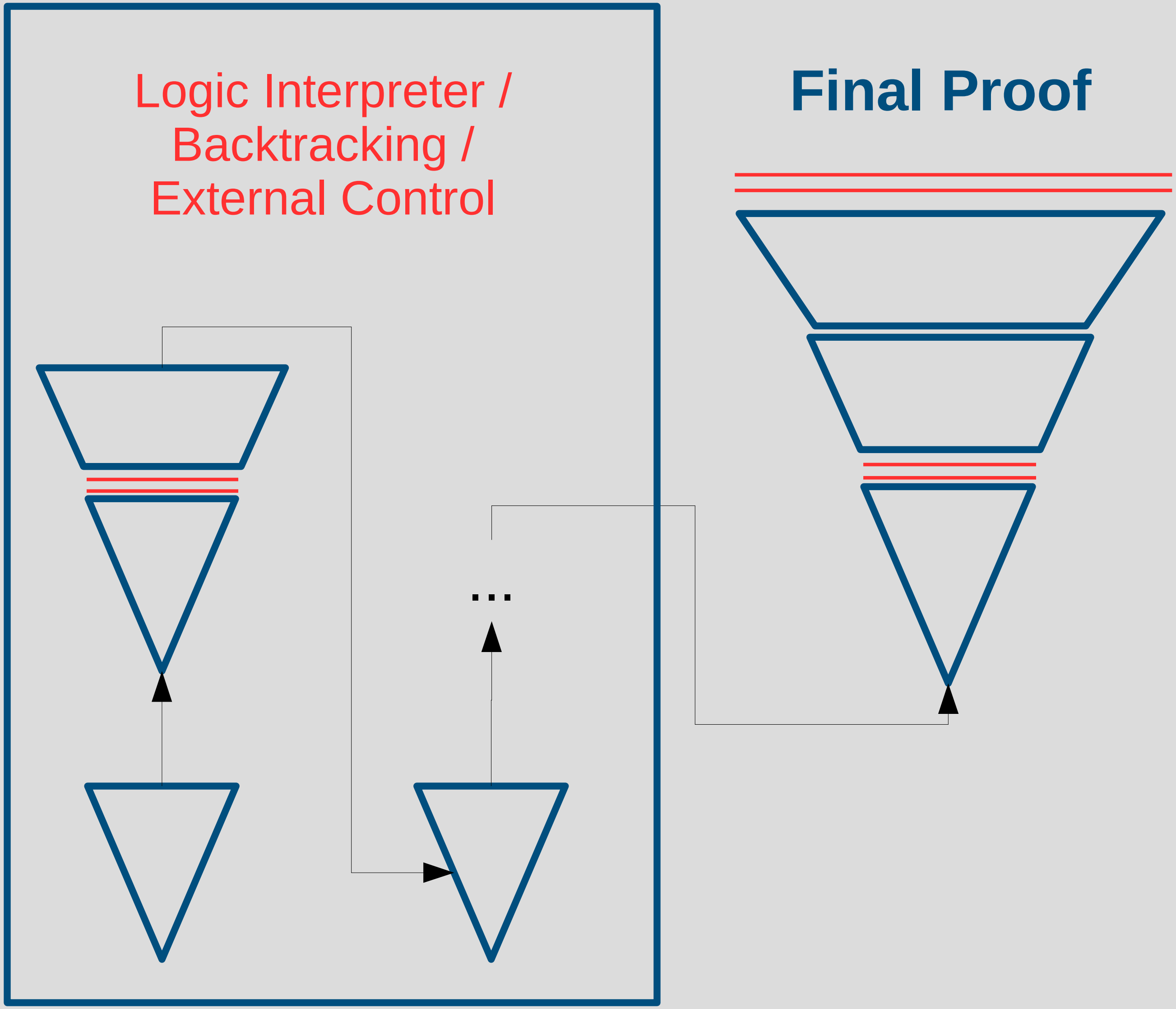
Focused proof search yields algorithmic specification



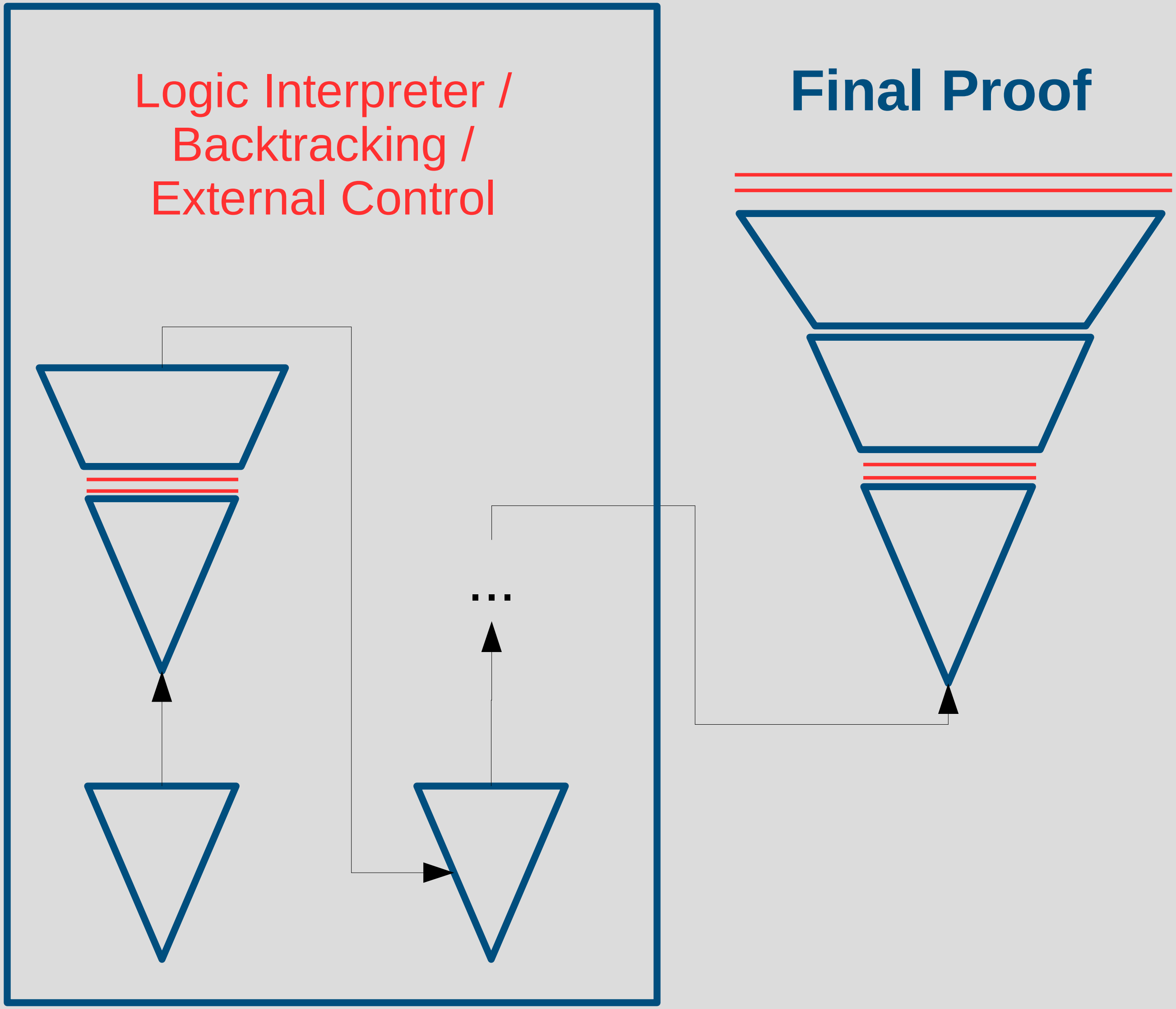
Final Proof



Focused proof search yields algorithmic specification



Focused proof search yields algorithmic specification



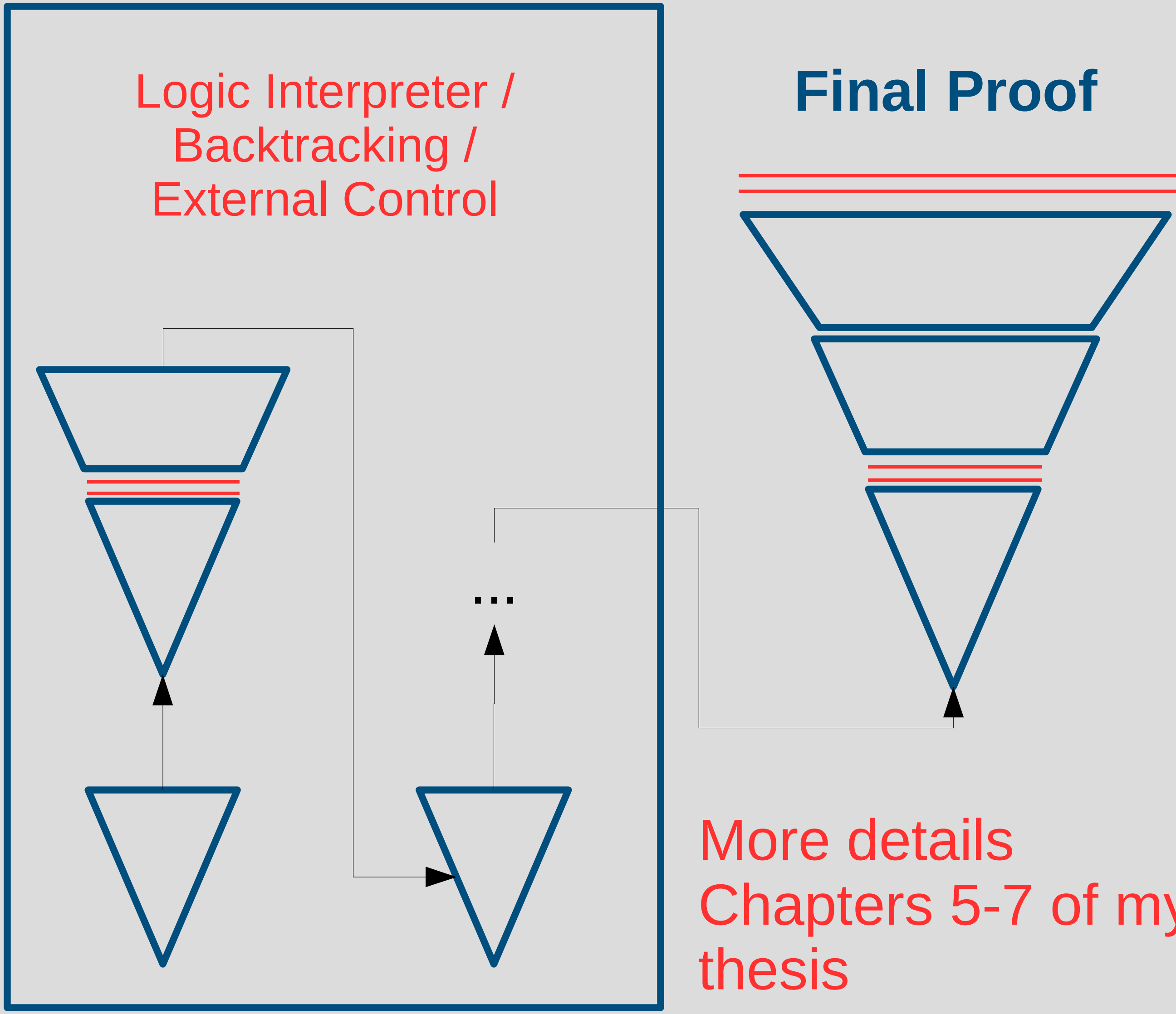
Focused proof search yields algorithmic specification

- (partial) computation runs are in 1-1 correspondence to (open) focused derivations: non-determinism in both can be made to match exactly.

- No external interpreter required: The proof theory of focused proof search is sophisticated enough to provide what is needed.

Controlling the size of focusing phases seems to be the key observation: global choice and local choice operators

Focused proof search yields algorithmic specification



More details Chapters 5-7 of my thesis

Focused proof search yields algorithmic specification

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Controlling the size of focusing phases seems to be the key observation: global choice and local choice operators

Agenda

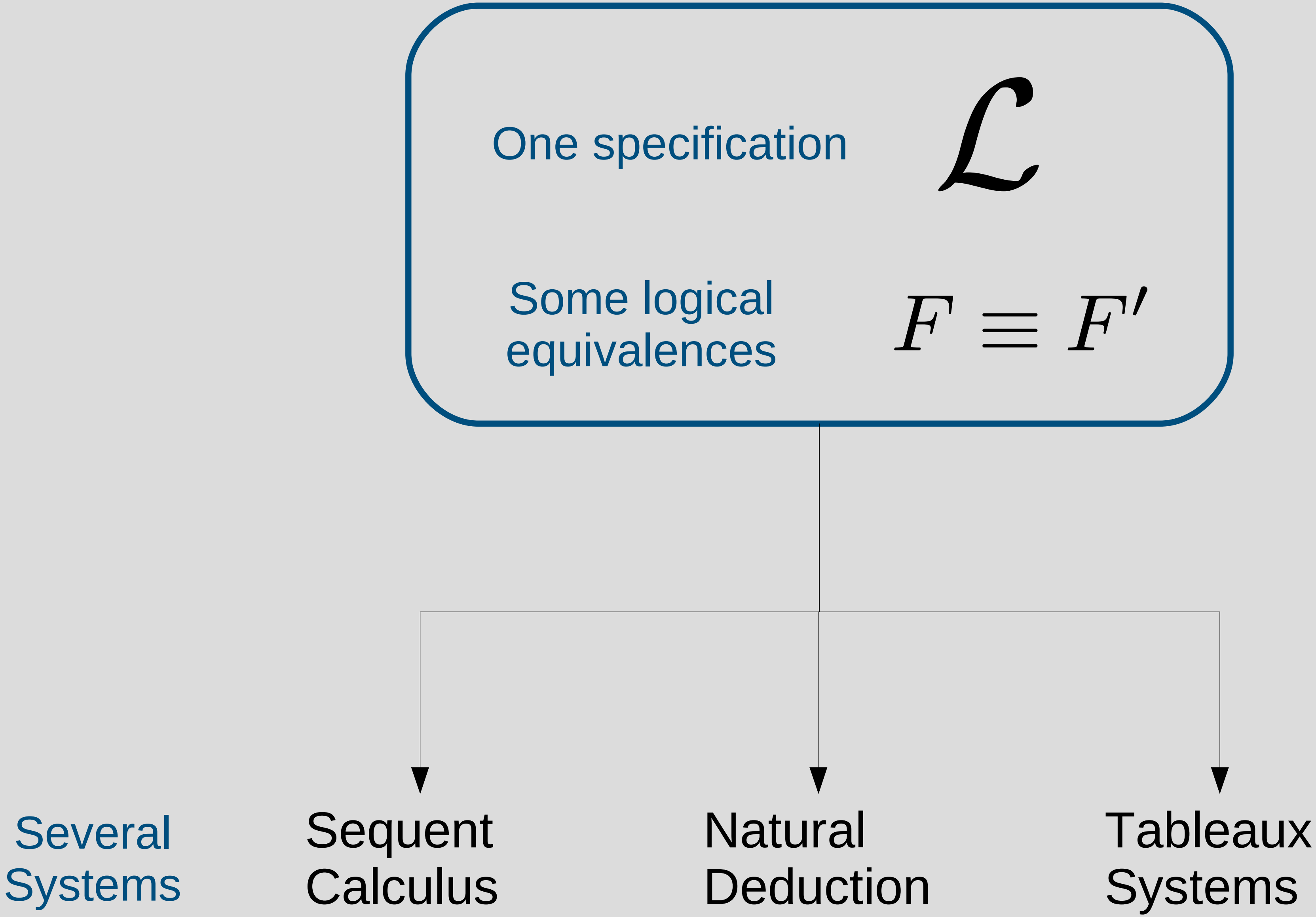
- Sequent Calculus
- Focusing
- Tabled Deduction
- Algorithmic Specifications
- **Logical Frameworks**

Overview

One specification $\mathcal{L}$

Some logical equivalences $F \equiv F'$

Overview


One specification
$\mathcal{L}$
Some logical equivalences
$F \equiv F'$
Several Systems
Sequent Calculus
Natural Deduction
Tableaux Systems

Encoding Logics

We consider only (first-order) minimal, intuitionistic and classical object logics.

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Encoding Formulas

OL

- Sequent Calculus – Left / Right
- Natural Deduction – Hyp / Con
- Tableaux – Neg / Pos

ML

$\lfloor \cdot \rfloor$ $\lceil \cdot \rceil$ form $\rightarrow o$

Encoding Logics

We consider only (first-order) minimal, intuitionistic and classical object logics.

	Encoding Formulas	Encoding Sequents
OL	• Sequent Calculus – Left / Right • Natural Deduction – Hyp / Con • Tableaux – Neg / Pos	$B_1, \ldots, B_n \vdash C_1, \ldots, C_m$
ML	$\lfloor \cdot \rfloor \quad \lceil \cdot \rceil \quad \text{form} \rightarrow o$	$\vdash \lfloor B_1 \rfloor, \ldots, \lfloor B_n \rfloor, \lceil C_1 \rceil, \ldots, \lceil C_m \rceil$

Theory $\mathcal{L}$ with the meaning of connectives – Existential Closure of

$(\Rightarrow_L)$	$\lfloor A \Rightarrow B \rfloor^\perp \otimes (\lceil A \rceil \otimes \lfloor B \rfloor)$	$(\Rightarrow_R)$	$\lceil A \Rightarrow B \rceil^\perp \otimes (\lfloor A \rfloor \parr \lceil B \rceil)$
$(\wedge_L)$	$\lfloor A \wedge B \rfloor^\perp \otimes (\lfloor A \rfloor \oplus \lfloor B \rfloor)$	$(\wedge_R)$	$\lceil A \wedge B \rceil^\perp \otimes (\lceil A \rceil \,\&\, \lceil B \rceil)$
$(\forall_L)$	$\lfloor \forall B \rfloor^\perp \otimes \lfloor Bx \rfloor$	$(\forall_R)$	$\lceil \forall B \rceil^\perp \otimes \forall x \lceil Bx \rceil$
$(\perp_L)$	$\lfloor \perp \rfloor^\perp$	(t_R)	$\lceil t \rceil^\perp \otimes \top$

Theory $\mathcal{L}$ with the meaning of connectives – Existential Closure of

$(\Rightarrow_L)$	$\lfloor A \Rightarrow B \rfloor^\perp \otimes (\lceil A \rceil \otimes \lfloor B \rfloor)$	$(\Rightarrow_R)$	$\lceil A \Rightarrow B \rceil^\perp \otimes (\lfloor A \rfloor \parr \lceil B \rceil)$
$(\wedge_L)$	$\lfloor A \wedge B \rfloor^\perp \otimes (\lfloor A \rfloor \oplus \lfloor B \rfloor)$	$(\wedge_R)$	$\lceil A \wedge B \rceil^\perp \otimes (\lceil A \rceil \,\&\, \lceil B \rceil)$
$(\forall_L)$	$\lfloor \forall B \rfloor^\perp \otimes \lfloor Bx \rfloor$	$(\forall_R)$	$\lceil \forall B \rceil^\perp \otimes \forall x \lceil Bx \rceil$
$(\perp_L)$	$\lfloor \perp \rfloor^\perp$	(t_R)	$\lceil t \rceil^\perp \otimes \top$

and the structural and identity rules

(Id_1)	$\lfloor B \rfloor^\perp \otimes \lceil B \rceil^\perp$	(Id_2)	$\lfloor B \rfloor \otimes \lceil B \rceil$
(Str_L)	$\lfloor B \rfloor^\perp \otimes ?\lfloor B \rfloor$	(Str_R)	$\lceil B \rceil^\perp \otimes ?\lceil B \rceil$
(W_R)	$\lceil C \rceil^\perp \otimes \perp$		

Theory $\mathcal{L}$ with the meaning of connectives – Existential Closure of

$$
\begin{array}{llll}
(\Rightarrow_L) & \lfloor A \Rightarrow B \rfloor^\perp \otimes (\lceil A \rceil \otimes \lfloor B \rfloor) & (\Rightarrow_R) & \lceil A \Rightarrow B \rceil^\perp \otimes (\lfloor A \rfloor \parr \lceil B \rceil) \\
(\wedge_L) & \lfloor A \wedge B \rfloor^\perp \otimes (\lfloor A \rfloor \oplus \lfloor B \rfloor) & (\wedge_R) & \lceil A \wedge B \rceil^\perp \otimes (\lceil A \rceil \,\&\, \lceil B \rceil) \\
(\forall_L) & \lfloor \forall B \rfloor^\perp \otimes \lfloor Bx \rfloor & (\forall_R) & \lceil \forall B \rceil^\perp \otimes \forall x \lceil Bx \rceil \\
(\perp_L) & \lfloor \perp \rfloor^\perp & (t_R) & \lceil t \rceil^\perp \otimes \top
\end{array}
$$

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$$
\begin{array}{llll}
(\mathsf{Id}_1) & \lfloor B \rfloor^\perp \otimes \lceil B \rceil^\perp & (\mathsf{Id}_2) & \lfloor B \rfloor \otimes \lceil B \rceil \\
(\mathsf{Str}_L) & \lfloor B \rfloor^\perp \otimes ?\lfloor B \rfloor & (\mathsf{Str}_R) & \lceil B \rceil^\perp \otimes ?\lceil B \rceil \\
(W_R) & \lceil C \rceil^\perp \otimes \perp & &
\end{array}
$$

$$
(\Rightarrow'_L) \quad \lfloor A \Rightarrow B \rfloor^\perp \otimes (!\lceil A \rceil \otimes \lfloor B \rfloor) \qquad (\mathsf{Id}'_2) \quad \lfloor B \rfloor \otimes !\lceil B \rceil
$$

Duality of the $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ atoms

$$\vdash \forall B(\lceil B\rceil \equiv \lfloor B\rfloor^{\perp})\,\&\,\forall B(\lfloor B\rfloor \equiv \lceil B\rceil^{\perp}), \mathrm{Id}_1, \mathrm{Id}_2$$

with Str_L and Str_R we prove the equivalences:

$$\lfloor B\rfloor \equiv ?\lfloor B\rfloor \text{ and } \lceil B\rceil \equiv ?\lceil B\rceil$$

Levels of Adequacy

We identify three levels of adequacy:

- Relative completeness: comparisons deal only with provability: the two systems have the same theorems.
- Full completeness of proofs: comparisons deal with proof objects: the proofs of a given formula are in one-to-one correspondence with proofs in another system.
- Full completeness of derivations: comparisons deal with derivations (*i.e.*, open proofs, such as inference rules themselves): the derivations in one system are in one-to-one correspondence with those in another system.

Levels of Adequacy

We identify three levels of adequacy:

- Relative completeness: comparisons deal only with provability: the two systems have the same theorems.
- Full completeness of proofs: comparisons deal with proof objects: the proofs of a given formula are in one-to-one correspondence with proofs in another system.
- Full completeness of derivations: comparisons deal with derivations (*i.e.*, open proofs, such as inference rules themselves): the derivations in one system are in one-to-one correspondence with those in another system.

We always obtain adequacy on the **level of derivations.**

Sequent Calculus

Sequent Calculus

if all $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ (meta-level) atoms are
negative

1) $\Gamma \vdash_{lm} C$ iff $\vdash \mathcal{L}_{lm}, \lfloor \Gamma \rfloor : \lceil C \rceil \Uparrow$
2) $\Gamma \vdash_{lj} C$ iff $\vdash \mathcal{L}_{lj}, \lfloor \Gamma \rfloor : \lceil C \rceil \Uparrow$
3) $\Gamma \vdash_{lk} \Delta$ iff $\vdash \mathcal{L}_{lk}, \lfloor \Gamma \rfloor, \lceil \Delta \rceil :\Uparrow$

Sequent Calculus

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3) $\Gamma \vdash_{lk} \Delta$ iff $\vdash \mathcal{L}_{lk}, \lfloor\Gamma\rfloor, \lceil \Delta \rceil :\Uparrow$

$$\mathcal{L}_{lk} = \mathcal{L} \cup \{\mathsf{Id}_1, \mathsf{Id}_2, \mathsf{Str}_L, \mathsf{Str}_R\},$$
$$\mathcal{L}_{lm} = \mathcal{L} \cup \{\mathsf{Id}_1, \mathsf{Id}_2', \mathsf{Str}_L, \Rightarrow_L'\} \setminus \{\bot_L, \Rightarrow_L\},$$
$$\mathcal{L}_{lj} = \mathcal{L} \cup \{\mathsf{Id}_1, \mathsf{Id}_2', \mathsf{Str}_L, \Rightarrow_L', W_R\} \setminus \{\Rightarrow_L\},$$

Sequent Calculus

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$$\mathcal{L}_{lk} = \mathcal{L} \cup \{\mathrm{Id}_1, \mathrm{Id}_2, \mathsf{Str}_L, \mathsf{Str}_R\},$$
$$\mathcal{L}_{lm} = \mathcal{L} \cup \{\mathrm{Id}_1, \mathrm{Id}'_2, \mathsf{Str}_L, \Rightarrow'_L\} \setminus \{\bot_L, \Rightarrow_L\},$$
$$\mathcal{L}_{lj} = \mathcal{L} \cup \{\mathrm{Id}_1, \mathrm{Id}'_2, \mathsf{Str}_L, \Rightarrow'_L, W_R\} \setminus \{\Rightarrow_L\},$$

We can also obtain a adequacy up to the level of derivations. For intuitionistic and minimal logics the ! is important.

The encoding of an inference rule (remember all meta-level atoms are negative):

$$\frac{\Gamma, A \Rightarrow B \vdash A \qquad \Gamma, A \Rightarrow B, B \vdash C}{\Gamma, A \Rightarrow B \vdash C}$$

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$$\frac{\Gamma, A \Rightarrow B \vdash A \qquad \Gamma, A \Rightarrow B, B \vdash C}{\Gamma, A \Rightarrow B \vdash C}$$

$$\dfrac{\dfrac{\dfrac{}{\vdash \mathcal{K} :\Downarrow \lfloor A \Rightarrow B \rfloor^{\perp}}\ [I_2] \qquad \dfrac{\dfrac{\vdash \mathcal{K} : \lceil A \rceil \Uparrow}{\vdash \mathcal{K} :\Downarrow !\lceil A \rceil}\ [!, R\Uparrow] \qquad \dfrac{\vdash \mathcal{K} : \lfloor B \rfloor, \lceil C \rceil \Uparrow}{\vdash \mathcal{K} : \lceil C \rceil \Downarrow \lfloor B \rfloor}\ [R\Downarrow, R\Uparrow]}{\vdash \mathcal{K} : \lceil C \rceil \Downarrow !\lceil A \rceil \otimes \lfloor B \rfloor}\ [\otimes]}{\vdash \mathcal{K} : \lceil C \rceil \Downarrow F}\ [2 \times \exists, \otimes]}{\vdash \mathcal{K} : \lceil C \rceil \Uparrow \cdot}\ [D_2]$$

F is $\exists A \exists B \lfloor A \Rightarrow B \rfloor^{\perp} \otimes (\lceil A \rceil \otimes \lfloor B \rfloor)$

The encoding of an inference rule (remember all meta-level atoms are negative):

$$\frac{\Gamma, A \Rightarrow B \vdash A \qquad \Gamma, A \Rightarrow B, B \vdash C}{\Gamma, A \Rightarrow B \vdash C}$$

$\lfloor A \Rightarrow B \rfloor \in \mathcal{K}$
is enforced

$$\cfrac{\cfrac{\cfrac{}{\vdash \mathcal{K} :\Downarrow \lfloor A \Rightarrow B \rfloor^{\perp}}\;[I_2] \qquad \cfrac{\cfrac{\vdash \mathcal{K} : \lceil A \rceil \Uparrow}{\vdash \mathcal{K} :\Downarrow !\lceil A \rceil}\;[!, R\Uparrow] \qquad \cfrac{\vdash \mathcal{K} : \lfloor B \rfloor, \lceil C \rceil \Uparrow}{\vdash \mathcal{K} : \lceil C \rceil \Downarrow \lfloor B \rfloor}\;[R\Downarrow, R\Uparrow]}{\vdash \mathcal{K} : \lceil C \rceil \Downarrow !\lceil A \rceil \otimes \lfloor B \rfloor}\;[\otimes]}{\vdash \mathcal{K} : \lceil C \rceil \Downarrow F}\;[2 \times \exists, \otimes]}{\vdash \mathcal{K} : \lceil C \rceil \Uparrow \cdot}\;[D_2]$$

F is $\exists A \exists B \lfloor A \Rightarrow B \rfloor^{\perp} \otimes (\lceil A \rceil \otimes \lfloor B \rfloor)$

The encoding of an inference rule (remember all meta-level atoms are negative):

$$\dfrac{\Gamma, A \Rightarrow B \vdash A \qquad \Gamma, A \Rightarrow B, B \vdash C}{\Gamma, A \Rightarrow B \vdash C}$$

$\lceil C \rceil$ must go to the right branch

$\lfloor A \Rightarrow B \rfloor \in \mathcal{K}$ is enforced

$$\dfrac{\dfrac{\dfrac{}{\vdash \mathcal{K} :\Downarrow \lfloor A \Rightarrow B \rfloor^{\perp}}\,[I_2] \qquad \dfrac{\dfrac{\vdash \mathcal{K} : \lceil A \rceil \Uparrow}{\vdash \mathcal{K} :\Downarrow\, !\lceil A \rceil}\,[!, R\Uparrow] \qquad \dfrac{\vdash \mathcal{K} : \lfloor B \rfloor, \lceil C \rceil \Uparrow}{\vdash \mathcal{K} : \lceil C \rceil \Downarrow \lfloor B \rfloor}\,[R\Downarrow, R\Uparrow]}{\vdash \mathcal{K} : \lceil C \rceil \Downarrow\, !\lceil A \rceil \otimes \lfloor B \rfloor}\,[\otimes]}{\vdash \mathcal{K} : \lceil C \rceil \Downarrow F}\,[2 \times \exists, \otimes]}{\vdash \mathcal{K} : \lceil C \rceil \Uparrow \cdot}\,[D_2]$$

F is $\exists A \exists B \lfloor A \Rightarrow B \rfloor^{\perp} \otimes (\lceil A \rceil \otimes \lfloor B \rfloor)$

Cut free proofs – remove the clause (ID_2) from the theory:

Cut free proofs – remove the clause (ID_2) from the theory:

if all $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ (meta-level) atoms are negative

1) $\Gamma \vdash^{f}_{lm} C$ iff $\vdash \mathcal{L}^{f}_{lm}, \lfloor \Gamma \rfloor : \lceil C \rceil \Uparrow$

2) $\Gamma \vdash^{f}_{lj} C$ iff $\vdash \mathcal{L}^{f}_{lj}, \lfloor \Gamma \rfloor : \lceil C \rceil \Uparrow$

3) $\Gamma \vdash^{f}_{lk} \Delta$ iff $\vdash \mathcal{L}^{f}_{lk}, \lfloor \Gamma \rfloor, \lceil \Delta \rceil :\Uparrow$

It is possible to obtain an adequacy on the level of derivations.

Natural Deduction [Sieg, Byrnes, 1998]

$$\frac{}{\Gamma, A \vdash_{nd} A \downarrow}\;[\mathrm{I}] \qquad \frac{\Gamma \vdash_{nd} F \uparrow \quad \Gamma \vdash_{nd} G \uparrow}{\Gamma \vdash_{nd} F \wedge G \uparrow}\;[\wedge I] \qquad \frac{\Gamma \vdash_{nd} F \wedge G \downarrow}{\Gamma \vdash_{nd} F \downarrow}\;[\wedge E]$$

$$\frac{\Gamma, A \vdash_{nd} B \uparrow}{\Gamma \vdash_{nd} A \Rightarrow B \uparrow}\;[\Rightarrow I] \qquad \frac{\Gamma \vdash_{nd} A \Rightarrow B \downarrow \quad \Gamma \vdash_{nd} A \uparrow}{\Gamma \vdash_{nd} B \downarrow}\;[\Rightarrow E] \qquad \frac{}{\Gamma \vdash_{nd} t \uparrow}\;[tI]$$

$$\frac{\Gamma \vdash_{nd} A\{c/x\} \uparrow}{\Gamma \vdash_{nd} \forall x\, A \uparrow}\;[\forall I] \qquad \frac{\Gamma \vdash_{nd} \forall x\, A \downarrow}{\Gamma \vdash_{nd} A\{t/x\} \downarrow}\;[\forall E] \qquad \frac{\Gamma \vdash_{nd} A \downarrow}{\Gamma \vdash_{nd} A \uparrow}\;[\mathrm{M}] \qquad \frac{\Gamma \vdash_{nd} A \uparrow}{\Gamma \vdash_{nd} A \downarrow}\;[\mathrm{S}]$$

Natural Deduction [Sieg, Byrnes, 1998]

$$\frac{}{\Gamma, A \vdash_{nd} A \downarrow}\ [\mathrm{I}] \quad \frac{\Gamma \vdash_{nd} F \uparrow \quad \Gamma \vdash_{nd} G \uparrow}{\Gamma \vdash_{nd} F \wedge G \uparrow}\ [\wedge I] \quad \frac{\Gamma \vdash_{nd} F \wedge G \downarrow}{\Gamma \vdash_{nd} F \downarrow}\ [\wedge E]$$

$$\frac{\Gamma, A \vdash_{nd} B \uparrow}{\Gamma \vdash_{nd} A \Rightarrow B \uparrow}\ [\Rightarrow I] \quad \frac{\Gamma \vdash_{nd} A \Rightarrow B \downarrow \quad \Gamma \vdash_{nd} A \uparrow}{\Gamma \vdash_{nd} B \downarrow}\ [\Rightarrow E] \quad \frac{}{\Gamma \vdash_{nd} t \uparrow}\ [tI]$$

$$\frac{\Gamma \vdash_{nd} A\{c/x\} \uparrow}{\Gamma \vdash_{nd} \forall x\, A \uparrow}\ [\forall I] \quad \frac{\Gamma \vdash_{nd} \forall x\, A \downarrow}{\Gamma \vdash_{nd} A\{t/x\} \downarrow}\ [\forall E] \quad \frac{\Gamma \vdash_{nd} A \downarrow}{\Gamma \vdash_{nd} A \uparrow}\ [\mathrm{M}] \quad \frac{\Gamma \vdash_{nd} A \uparrow}{\Gamma \vdash_{nd} A \downarrow}\ [\mathrm{S}]$$

$\Gamma \vdash_{nd} C \uparrow$ $\qquad \vdash \Sigma, \lfloor \Gamma \rfloor : \lceil C \rceil \Uparrow$

Useful to identify normal proofs, where the S rules is not allowed.

$\Gamma \vdash_{nd} C \downarrow$ $\qquad \vdash \Sigma, \lfloor \Gamma \rfloor : \lfloor C \rfloor^{\perp} \Uparrow$

Natural Deduction – including normal forms

if all $\lceil\cdot\rceil$ (meta-level) atoms are negative
if all $\lfloor\cdot\rfloor$ (meta-level) atoms are positive

1) $\Gamma \vdash_{nj} C\uparrow$ iff $\vdash \mathcal{L}_{lj}, \lfloor\Gamma\rfloor : \lceil C\rceil \Uparrow$
2) $\Gamma \vdash^{n}_{nj} C\uparrow$ iff $\vdash \mathcal{L}^{f}_{lj}, \lfloor\Gamma\rfloor : \lceil C\rceil \Uparrow$
3) $\Gamma \vdash^{n}_{nj} C\downarrow$ iff $\vdash \mathcal{L}^{f}_{lj}, \lfloor\Gamma\rfloor : \lfloor C\rfloor^{\perp} \Uparrow$

An adequacy on the level of derivations can also be obtained.

Natural Deduction – including normal forms

if all $\lceil\cdot\rceil$ (meta-level) atoms are negative
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An adequacy on the level of derivations can also be obtained.

Since the polarity assignment a focused system does not affect provability, we obtain the following relative completeness result for free:

Corollary

$$\Gamma \vdash_{lj} C \text{ iff } \Gamma \vdash_{nj} C \quad \text{and} \quad \Gamma \vdash^{f}_{lj} C \text{ iff } \Gamma \vdash^{n}_{nj} C.$$

Cut now becomes Switch Rule:

$$\frac{\Gamma \vdash_{nd} C \uparrow}{\Gamma \vdash_{nd} C \downarrow}\ [S]$$

$$\dfrac{\dfrac{}{\vdash \Sigma, \lfloor\Gamma\rfloor : \lfloor C\rfloor^{\perp} \Downarrow \lfloor C\rfloor}\ [I_1] \quad \dfrac{\vdash \Sigma, \lfloor\Gamma\rfloor : \lceil C\rceil \Uparrow}{\vdash \Sigma, \lfloor\Gamma\rfloor :\Downarrow \lceil C\rceil}\ [R\Downarrow, R\Uparrow]}{\dfrac{\vdash \Sigma, \lfloor\Gamma\rfloor : \lfloor C\rfloor^{\perp} \Downarrow \lfloor C\rfloor \otimes \lceil C\rceil}{\vdash \Sigma, \lfloor\Gamma\rfloor : \lfloor C\rfloor^{\perp} \Uparrow}\ [D_2, \exists]}\ [\otimes]$$

Other proof systems

In Chapter 4, we also deal with:

- Systems with generalized elimination and introduction rules
- the KE tableaux of D'Agostino and Mondadori, and
- a proof system of Smullyan with many axioms and with cut as the only inference rule.

My Thesis

Exploiting non-canonicity in the Sequent Calculus

- Polarity assignment of literals in focused systems
- Linear logic's exponentials

- Tabled deduction
- Logical frameworks
- Algorithmic specifications